THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, April 28, 1900. The American Physical Society was in session at the same time, and a part of the afternoon was devoted to a joint meeting, at which the papers noted in the list below were presented by Professors E. W. Brown and R. S. Woodward. The attendance during the day was about forty, including the following twenty-four members of the Society:

Professor E. W. Brown, Professor F. N. Cole, Dr. W. S. Dennett, Professor T. S. Fiske, Mr. A. S. Gale, Dr. G. B. Germann, Dr. A. A. Himowich, Mr. S. A. Joffe, Mr. C. J. Keyser, Dr. James Maclay, Dr. Emilie N. Martin, Dr. G. A. Miller, Mr. C. A. Noble, Professor M. I. Pupin, Professor J. K. Rees, Mr. C. H. Rockwell, Professor Charlotte A. Scott, Dr. Virgil Snyder, Professor E. B. Van Vleck, Professor L. A. Wait, Professor A. G. Webster, Dr. J. Westlund, Miss E. C. Williams, Professor R. S. Woodward.

The President of the Society, Professor R. S. Woodward, occupied the chair, yielding it during the joint meeting to Professor William Hallock of the Physical Society. The amendments of the Constitution outlined in the report of the February meeting were adopted. The Council announced the election of the following persons to membership in the Society: Professor R. D. Ford, St. Lawrence University, Canton, N. Y.; Dr. L. W. Reid, Princeton University, Princeton, N. J. Eight applications for membership were reported.

The following papers were read at this meeting:

(1) Dr. VIRGIL SNYDER: "On some invariant scrolls in collineations which leave a group of five points invariant."

(2) Mr. A. S. GALE: "Note on four theorems of Chasles."

(3) Professor CHARLOTTE A. SCOTT: "A theorem on quadrilaterals in space" (preliminary communication).

(4) Mr. F. H. LOUD: "Sundry theorems concerning n-lines in a plane."

(5) Dr. E. J. WILCZYNSKI: "Transformation of systems of linear differential equations."

(6) Professor FLORIAN CAJORI: "Semi-convergent and divergent series whose product is absolutely convergent."

(7) Professor E. W. BROWN: "A possible explanation of the eleven year period of sunspot activity."

(8) Professor R. S. WOODWARD: "An elementary method of integrating certain linear differential equations."

(9) Dr. G. A. MILLER: "On a certain class of abelian groups."

(10) Professor H. B. NEWSON: "On singular transformations and continuous groups."

(11) Professor E. O. LOVETT: "Group theory and geometry of four dimensions."

(12) Professor E. O. LOVETT: "The condition that a linear total differential equation be integrable."

(13) Professor C. H. HINTON: "Observations on the principle of duality."

Professor Hinton's paper was presented to the Society through Professor Lovett. In the absence of the authors, Mr. Loud's paper was read by Professor C. A. Scott, and the papers of Dr. Wilczynski, Professor Cajori, Professor Newson, Professor Lovett, and Professor Hinton were read by title. Abstracts of the papers are given below.

Dr. Snyder's paper is in abstract as follows: There are six cyclical collineations of space whose orders are less than seven: T_2, the harmonic homology; T_3, the axial involution; $T_4 \equiv (123)(4)(5)$, in which the elements of space are in triads; $T_5 \equiv (123)(45)$, in sextuples; $T_6 \equiv (1234)(5)$, in quadruples; and $T_7 \equiv (12345)$, in quintuples. A cyclical collineation of order six and having five elements chosen arbitrarily does not exist. Systems of scrolls contained in a linear congruence which are left invariant by these transformations are discussed, with particular reference to their asymptotic lines. The transformations T_3, T_4, and T_7 leave a [3, 1] scroll invariant in such a way that each asymptotic line is transformed into itself. The quartic surface, locus of the vertex of a quadric cone which passes through two triads of T_4, goes into itself; it touches a [3, 1] scroll in every point of the twisted cubic through the six points. This cubic is an asymptotic line on both surfaces. Every [3, 1] scroll having a pair of double imaginary pinch points has asymptotic lines of order 3.

In 1860 Chasles published the theorems that, if a right line be displaced, the middle points of the chords joining congruent points lie on a line or are identical, and that the planes perpendicular to these chords at their middle points pass through a line or are identical; and also corresponding theorems relating to the displacement of a plane figure.

These theorems form the basis of the geometrical consideration of the theory of finite displacements in space as developed by Study. Mr. Gale's note, which will appear in the *Annals of Mathematics*, shows how the theorems of Chasles follow from Wiener's theorem that every displacement in space may be resolved in ∞^2 ways into the product of two axis reflections. By means of this theorem, displacements are resolved into the product of two transformations such that the points of an arbitrary line or plane are invariant with respect to the first transformation. The theorems of Chasles follow immediately from properties of the second transformation.

The following is a summary of Professor Scott's preliminary communication: If corresponding vertices a, a'; b, b'; c, c'; d, d'; e, e'; f, f' of two quadrilaterals in different planes are joined, an interesting configuration of 6 lines $\alpha, \beta, \gamma, \delta, \epsilon, \varphi$ is obtained. It is cut by a singly infinite system of planes in complete quadrilaterals; these planes, being common tangent planes to two ruled quadrics with a common generator, are tangent planes to a cubic torse, and consequently also to a twisted cubic. The configuration is intimately associated with the theory of cubic curves and torses, yet there seems to be no recognition of the theorem in the literature of this subject. There is no difficulty in obtaining a proof in various forms; but to exhibit the theorem as fundamental in the treatment of projective forms, a proof that shall rely solely on the formal properties of points, lines, and planes is desirable. It appears that such a proof must be obtainable by means of triangles in perspective; but it still remains to be found.

The point of departure for Mr. Loud's paper is furnished by Professor Morley's article in the April *Transactions* "On the metric geometry of the plane n-line." Professor Morley, having discussed a certain class of theorems resulting from combining n-lines by twos, had pointed out that they might also be combined three or more at a time with analogous results. The present paper considers the case of combination by threes. After briefly noting those of his results which are strictly analogous to some of Professor Morley's, the author has indicated a means of deducing one series of theorems from the other. A distinction between the two series is pointed out in the fact that in the present case each line is restricted to a single direction. An inquiry into the relation of two figures, derived one from the other by the

reversal of one given line, follows, and occupies the greater part of the paper. The general nature of the theorems discussed is exemplified in the following: The theorem which in the parallelism of the two series becomes the analogue of Miquel's asserts that the twenty points, which are centers of circles tangent to six given directed lines taken by threes, lie by fours on fifteen circles, whose circumferences meet by fives in six points, and these points lie on a single circle. Again, as an illustration of the effect of reversal, we have the theorem: If all but one of an even number of given lines be turned, each through a right angle, about the point at which it meets the one remaining line, and if the Clifford circles be constructed for these rotated lines in both their original position and that to which they are turned, these circles intersect orthogonally. By making each line in succession the one upon which the others are turned, as many orthogonal pairs of circles are obtained as there are given lines, and all these circles pass through a common point, which is the focus of the p-fold parabola tangent to all the given lines.

Dr. Wilczynski's paper furnishes a proof of the following theorem: The most general transformation which converts a general system of linear homogenous differential equations into another system of the same form and order is

$$x = f(\xi), \quad y_k = a_{k1}(\xi)\eta_1 + \cdots + a_{kn}(\xi)\eta_n \quad (k = 1, 2, \cdots, n),$$

where f and a_{ki} are arbitrary functions of ξ. This theorem is the basis of a theory of invariants of such systems upon which the author is now engaged.

Pringsheim states in one of his articles that he sees no *a priori* reason why one might not be able to construct semi-convergent series possessing properties which would render the product of two series absolutely convergent. But he was unable to produce examples of such series. In the present paper Professor Cajori gives not only pairs of semi-convergent series, but also pairs of series, one semi-convergent, the other divergent, whose product converges absolutely. The process of constructing the series is explained. In all cases brought under consideration, one of the factor series has zero for its sum. Example: Of the following two series, T_1 is semi-convergent, T_2 is divergent; their product is absolutely convergent:

$$T_1 \equiv \sum_{p=0}^{p=\infty}\left(\frac{1}{6p+1}-\frac{1}{6p+2}+\frac{1}{6p+3}-\frac{1}{6p+1}+\frac{1}{6p+2}-\frac{1}{6p+3}\right),$$

$$T_2 \equiv \sum_{p=0}^{p=\infty}\left(\frac{1}{6p+1}+\frac{1}{6p+2}-\frac{1}{6p+3}+\frac{1}{6p+1}+\frac{1}{6p+2}-\frac{1}{6p+3}\right).$$

The following differential equations are of frequent occurrence in mechanics and mathematical physics:

$$\frac{dx}{dt}+ay=0, \quad \frac{dy}{dt}-ax=0, \tag{1}$$

and

$$\frac{d^2x}{dt^2}+2n\frac{dx}{dt}+a^2x=0, \tag{2}$$

where a and n are constants. The integrals of (1) are well known to be

$$x=a\cos(at+\theta), \quad y=a\sin(at+\theta), \tag{3}$$

where a and θ are constants. The integral of (2) is well known to take different forms according as $n^2>a^2$, $n^2=a^2$, or $n^2<a^2$. Equation (2) represents the phenomena of damped vibrations. When $n=0$, it represents simple harmonic motion in a straight line; while equations (1) represent the rectangular components of uniform motion in a circle. The usual methods of integrating these equations require some knowledge of the theory of differential equations in general. It is the object of Professor Woodward's to show how to integrate them without the aid of such knowledge. To this end, observe that (1) and (2) are comprised in the following equations of the first order:

$$\frac{dx}{dt}+ay+2nx=0, \quad \frac{dy}{dt}-ax=0. \tag{4}$$

These equations give

$$\frac{dx}{2nx+ay}=-\frac{dy}{ax}=-dt,$$

whence, by the introduction of two factors β, γ,

$$\frac{\beta dx}{2nx+ay}=-\frac{\beta\gamma dy}{a\gamma x}=-\beta dt,$$

or

$$\frac{\beta dx-\beta\gamma dy}{(2n+a\gamma)x+ay}=-\beta dt. \tag{5}$$

The first member of this equation will be an exact differential if

$$\beta = 2n + \alpha\gamma, \quad \text{and} \quad -\beta\gamma = a, \tag{6}$$

whence

$$\alpha\gamma = -n \pm \sqrt{n^2 - a^2}.$$

Thus the two values of β derived from (6) are

$$\beta_1 = n + \sqrt{n^2 - a^2}, \quad \beta_2 = n - \sqrt{n^2 - a^2}. \tag{7}$$

Writing $\omega^2 = n^2 - a^2$, it is seen that

$$\begin{aligned} &\beta_1 = n + \omega, \quad \beta_2 = n - \omega && (n^2 > a^2)\ ; \\ &\beta_1 = n, \qquad \beta_2 = n && (n^2 = a^2)\ ; \\ &\beta_1 = n + i\omega, \quad \beta_2 = n - i\omega && (n^2 < a^2). \end{aligned} \tag{8}$$

Introducing β_1 and β_2 in (4), and observing the second of (6), there result

$$\frac{\beta_1 dx + a dy}{\beta_1 x + a y} = -\beta_1 dt, \quad \frac{\beta_2 dx + a dy}{\beta_2 x + a y} = -\beta_2 dt.$$

Integrating these and calling the values of x and y for $t = 0$, x_0 and y_0 respectively,

$$\begin{aligned} \beta_1 x + a y &= (\beta_1 x_0 + a y_0) e^{-\beta_1 t}, \\ \beta_2 x + a y &= (\beta_2 x_0 + a y_0) e^{-\beta_2 t}, \end{aligned}$$

whence by elimination

$$\begin{aligned} (\beta_2 - \beta_1) x &= (\beta_2 x_0 + a y_0) e^{-\beta_2 t} - (\beta_1 x_0 + a y_0) e^{-\beta_1 t}, \\ a(\beta_2 - \beta_1) y &= (\beta_1 \beta_2 x_0 + a\beta_2 y_0) e^{-\beta_1 t} - (\beta_1 \beta_2 x_0 + a\beta_1 y_0) e^{-\beta_2 t}. \end{aligned} \tag{9}$$

Substituting in these the first pair of values from (8), there result

$$\begin{aligned} x &= e^{-nt}\left(x_0 \cosh \omega t - \frac{n x_0 + a y_0}{\omega} \sinh \omega t\right), \\ y &= e^{-nt}\left(y_0 \cosh \omega t + \frac{a x_0 + n y_0}{\omega} \sinh \omega t\right), \end{aligned} \qquad (n^2 > a^2). \tag{10}$$

Similarly, using in (9) the last pair of values in (8), or replacing ω in (10) by $i\omega$,

$$\begin{aligned} x &= e^{-nt}\left(x_0 \cos \omega t - \frac{n x_0 + a y_0}{\omega} \sin \omega t\right), \\ y &= e^{-nt}\left(y_0 \cos \omega t + \frac{a x_0 + n y_0}{\omega} \sin \omega t\right), \end{aligned} \qquad (n^2 < a^2). \tag{11}$$

For the intermediate case in which $n^2 = a^2$, or $\omega = 0$, it is seen at once from (10) or (11) that

$$x = e^{-nt}\{x_0 - a(x_0 + y_0)t\},$$
$$y = e^{-nt}\{y_0 + a(x_0 + y_0)t\}, \qquad (n^2 = a^2). \qquad (12)$$

For the special case in which $n = 0$, and hence $\omega = a$, or for the case of simple harmonic motion specified by (1), (11) give

$$x = x_0 \cos at - y_0 \sin at, \qquad y = y_0 \cos at + x_0 \sin at,$$

which become identical with (3) by means of the substitution $x_0 = a \cos \theta$, $y_0 = a \sin \theta$.

The following is an abstract of Dr. Miller's paper: The $\varphi(g)$ numbers which are less than g and prime to g constitute a group with respect to multiplication if the smallest positive residues mod g are taken in place of the products. This group G_1 is the group of isomorphisms of the cyclical group of order g. When G_1 is cyclical its order g_1 must be $p^\alpha(p-1)$, where p is some odd prime number. From this it is clear that g_1 must be even whenever G_1 is cyclical. In fact g_1 is always even. If $g_1 = 2^\beta$, G cannot have more than two independent generators of the same order except for the special case when $g = 24$. In this case G_1 is clearly of type (1, 1, 1).

Professor Lovett's first paper is in summary as follows: Speculations relative to n-dimensional space have followed several fairly well-defined trends: 1° A direct extension of the cartesian geometry, which extension is to be regarded as nothing more than a convenient form of phraseology; in this form n-dimensional space sprang forth from the minds of Grassmann, Cayley, Gauss, and Cauchy, and the idea was likely familiar to Euler and Lagrange. 2° The transformation of the ordinary visualizable spaces of two and three dimensions into manifoldnesses of higher or lower dimensions by introducing space elements other than the point or its dual element; for example, the line geometry of Plücker, the sphere geometry of Lie, the five-dimensional manifoldness of all conics in the plane as an auxiliary to Ball's theory of screws; this category becomes more concrete perhaps than any other. 3° The absolute geometry of space; here would appear the famous dissertation of Riemann, the well-known memoirs of Helmholtz, Klein, and

Lie, and the elaborate treatise of Veronese. 4° The extension of the methods and results of ordinary differential geometry and analysis situs to spaces of many dimensions; to this class belongs the works of Christoffel, Beltrami, Bianchi, Cesàro, and Ricci, and the quite recent contributions of Darboux and his followers. 5° The direct extention of the problems and concepts of metrical and projective geometry of ordinary space, as exemplified in the memoirs of Jordan, d'Ovidio, and Veronese. 6° The theory of birational correspondences between n-dimensional aggregates as studied by Noether, Kantor, and Brill. 7° The descriptive geometry of space of n dimensions as begun in the papers of Veronese, Stringham, Schlegel, and Segre. 8° The kinematics of higher spaces as developed by Jordan, Clifford, and Beltrami. 9° The interpretation of n-dimensional geometry in the light of the theory of groups as exhibited by Klein, Lie, and Poincaré.

It is proposed in the present paper to make an expository contribution to the ninth and fourth of the above categories, constructing ordinary four-dimensional space by the method of Lie's theory of continuous groups and studying curves of triple curvature by the intrinsic analysis developed by Cesàro in his Lezioni di geometria intrinseca.

In Professor Lovett's second paper, which will appear in the *Annals of Mathematics*, the well-known property $(U, V) \equiv 0$ of the Jacobian system

$$Uf \equiv \frac{\partial f}{\partial x} + \zeta_1 \frac{\partial f}{\partial z} = 0, \qquad Vf \equiv \frac{\partial f}{\partial y} + \zeta_2 \frac{\partial f}{\partial z} = 0,$$

is employed to derive the ordinary criterion for the integrability of the equation

$$Pdx + Qdy + Rdz = 0.$$

The result extends itself immediately to the case of n variables.

Professor Hinton's paper investigates the principle of duality as it would exist in a fictitious geometrical world in which the distance of a point is measured by means of the difference, instead of the sum, of the squares of its abscissa and ordinate.

F. N. Cole.

Columbia University.

THE APRIL MEETING OF THE CHICAGO SECTION.

THE seventh regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at Northwestern University, Evanston, Ill., on Saturday, April 14, 1900. The morning session was called to order at ten o'clock, by Professor E. H. Moore, Vice-President of the Society.

The following members were present:

Professor E. W. Davis, Professor Thomas F. Holgate, Mr. H. G. Keppel, Dr. Kurt Laves, Professor H. Maschke, Professor E. H. Moore, Professor James B. Shaw, Professor E. B. Skinner, Dr. H. F. Stecker, Professor H. S. White, Professor J. W. A. Young.

Professor James B. Shaw was elected chairman for the meeting, which extended through a forenoon and an afternoon session.

Papers were presented as follows:

(1) Professor E. H. MOORE: "Evaluation of an important determinant of special form."

(2) Professor H. MASCHKE: "A new method of determining the differential parameters and invariants of quadratic differential quantics."

(3) Mr. ROBERT E. MORITZ: "On the transcendency of π."

(4) Dr. H. F. BLICHFELDT: "Note on the functions of a given class which in a given interval differ the least possible from zero."

(5) Dr. KURT LAVES: "Maupertuis's principle of least action for forces involving the coördinates and their first and higher differential quotients."

(6) Professor MARTIN KRAUSE: "On systems of differential equations which are satisfied by quadruply periodic functions."

(7) Professor JAMES B. SHAW: "Multiple algebra."

(8) Professor E. W. DAVIS: "A graphic method for the solution of systems of linear equations."

(9) Professor L. E. DICKSON: "Determination of an abstract group of order $2^7.3^4.5.7$ holoedrically isomorphic with a certain orthogonal group and with a certain hyperabelian group."

(10) Professor E. H. MOORE: "On the generational determination of abstract groups."

(11) Professor H. B. NEWSON: "On the group of 216 collineations of the plane."

(12) Mr. JAMES ARCHY SMITH: "Definition and periodicity of the abstract group holoedrically isomorphic with a certain group of algebraic transformations."

(13) Professor E. W. DAVIS: "On the use of color in the representation of crystal groups."

Mr. Moritz's paper was presented to the Society and read by Professor Davis; Dr. Blichfeldt's and Mr. Smith's papers were presented through Professor Moore, and the former was also read by Professor Moore; Professor Krause's paper was presented to the Society and read by Professor Maschke. Professor Newson's paper was read by title. Abstracts of the papers are given below.

Professor Moore's first paper, which has been offered for publication to the *Annals of Mathematics*, contains, after certain generalities on determinantal notation, four proofs of the theorem that the determinant

$$|\, a_{gi,\,hj} \,| \qquad \begin{pmatrix} g,\, h = 1, \cdots, m \\ i,\, j = 1, \cdots, n \end{pmatrix}$$

of order mn, where throughout

$$a_{gi,\,hj} = b^{(i)}_{gh}\, c^{(h)}_{ij},$$

is the continued product of the n determinants $B^{(i)}$ of order m and the m determinants $C^{(h)}$ of order n

$$B^{(i)} = |\, b^{(i)}_{gh} \,|, \quad C^{(h)} = |\, c^{(h)}_{ij} \,|.$$

This theorem, stated in a different notation, was first discovered by Professor W. H. Metzler.

Professor Maschke's paper presents a symbolic method for the treatment of differential parameters and differential invariants. If F and Φ are any two invariants (*i. e.*, differential parameters or differential invariants) of the binary quadratic differential quantic

$$A = \sum_{r,\,s=1}^{2} a_{rs}\, dx_r\, dx_s,$$

then the expression

$$\frac{1}{\sqrt{a_{11}a_{22} - a_{12}^2}} \begin{vmatrix} \dfrac{\partial F}{\partial x_1} & \dfrac{\partial F}{\partial x_2} \\ \dfrac{\partial \Phi}{\partial x_1} & \dfrac{\partial \Phi}{\partial x_2} \end{vmatrix} = \{F,\, \Phi\}$$

is again an invariant of A. Let $f, \varphi, \cdots, u, v, \cdots$ be arbitrary functions of x_1, x_2; as such they are invariants of A, and consequently the expressions

$$\{f, u\}, \quad \{f, \{\varphi, u\}\}, \quad \{\{f, u\}, \{\varphi, v\}\}, \quad \text{etc.}, \qquad (1)$$

will be invariants of A. If now we agree to set

$$\frac{\partial f}{\partial x_r} \cdot \frac{\partial f}{\partial x_s} = a_{rs},$$

we obtain

$$\left(\frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2\right)^2 = \Sigma a_{rs} dx_r dx_s = A,$$

and we call f a *symbol* of A. It follows then, as in algebra, that every product of expressions (1) containing each symbol $f, \varphi, \cdots$ precisely twice (with certain restrictions concerning the higher derivatives) will be an actual invariant of A. Thus the following invariants are obtained:

$$\{f, u\}^2 = \Delta_1 u, \quad \{f, u\}\{f, v\} = \nabla(u, v), \quad \{f, \{f, u\}\} = \Delta_2 u,$$
$$\{f, \varphi\}\{\{f, u\}, \{\varphi, u\}\} = 2\Delta_{22} u, \quad \{\varphi, \psi\}\{\{f, \varphi\}, \{f, \psi\}\} = 2K.$$

The extension of the method from the case of binary differential quantics to that of n variables, from quadratic quantics to those of higher orders, from invariants to covariants, and also from invariants and covariants of one quantic to those of two or more quantics, is obvious. The full paper will appear in the *Transactions*.

To prove that π is transcendental it is sufficient to show that e cannot satisfy an equation of the form

$$C_0 + e^{\alpha_1} + e^{\alpha_2} + \cdots + e^{\alpha_n} = 0, \qquad [A]$$

where C_0 is an integer and the α's are associated algebraic numbers. Hurwitz has shown that, for every rational function $f(x) = 0$, another function $F(x) = 0$ exists, such that

$$F(x) - e^x F(0) = -xe^{(1-\theta)x} f(\theta x) \quad (0 < \theta < 1).$$

Mr. Moritz points out that by putting

$$f(x) = \frac{x^{p-1}}{(p-1)!} \cdot a_0^{np-1} \cdot \prod_{i=1}^{n} (a_i - x)^p$$

the assumption [A] leads to the equation

$$C_0 F(0) + \sum_{i=1}^{n} F(a_i) = \sum_{i=1}^{n} [-a_i e^{(1-\theta)a_i} f(\theta a_i)],$$

in which $C_0F(0)$ is not divisible by the integer p, $\Sigma F(a_i)$ is divisible by p, and the right member of the equation approaches zero as a limit with increasing values of p. This is absurd; hence $[A]$ does not exist.

A real function of a real variable and certain real parameters being given, Tschebycheff has treated the problem of determining the parameters so that in a given interval the maximum absolute value of the function shall be as small as possible. In Dr. Blichfeldt's paper, which has been offered to the *Transactions* for publication, the necessary conditions on the parameters which Tschebycheff exhibited are found, when supplemented by certain others, to be also sufficient conditions.

Dr. Laves shows that the theorem of Maupertuis can be proved to exist for certain forces of the general nature indicated in the title of his paper which possess an effective potential W. The potential W is an arbitrary function of the coördinates and velocities of the bodies involved, not containing the time t explicitly. The differential equations of motion for effective potential forces are, in Lagrange's second form,

$$\frac{d}{dt}\left(\frac{\partial(T+W)}{\partial q_i'}\right)-\frac{\partial}{\partial q_i}(T+W)=0 \quad (i=1,\cdots,k),$$

where T is the kinetic energy of the system. Starting from Jacobi's partial differential equation of the first order

$$\frac{\partial v}{\partial t}-K\left(q_1,\cdots,q_k,\frac{\partial v}{\partial q_1},\cdots,\frac{\partial v}{\partial q_k}\right)=0,$$

we assume the complete integral

$$v=-ht+\bar{v}(q_1,\cdots,q_k,a_1,\cdots,a_{k-1})+a_k$$

and define the action of the system during the interval of time t_1-t_0 by

$$A=\int dv=\int_{t_0}^{t_1}\Sigma\frac{\partial\bar{v}}{\partial q_1}q_1'\,dt=\int_{t_0}^{t_1}\Sigma p_i q_i'\,dt$$

where

$$p_i=\frac{\partial(T+W)}{\partial q_1'}.$$

Calling $T+W=\varphi$, we make the special assumption that $\varphi=\varphi_0+\varphi_1+\varphi_2$, where φ_0, φ_1, φ_2 are homogeneous functions

of q_i' of the degree indicated by the subscript in each case. The integral of the conservation of vis viva is

$$K \equiv \Sigma q_i' \frac{\partial(T+W)}{\partial q_i'} - (T+W) \equiv \varphi_2 - \varphi_0 = h.$$

The expression for A becomes

$$A = \int_{t_0}^{t_1} (2\varphi_2 + \varphi_1) dt.$$

We may eliminate dt by means of the equation $\varphi_2 = \varphi_0 + h$ by considering

$$\varphi_2 = \Sigma a_{ij} \frac{dq_i dq_j}{dt^2} \equiv \frac{d\sigma^2}{dt^2},$$

where we have put $d\sigma^2 \equiv \Sigma a_{ij} dq_i dq_j$. Thus we obtain

$dt = \dfrac{d\sigma}{\sqrt{\varphi_0 + h}}$. Putting $2\varphi_2 = 2(\varphi_0 + h)$ and $\varphi_1 \equiv \Sigma \dfrac{b_i dq_i}{dt} \equiv \dfrac{d\tau}{dt}$ we obtain $A = \displaystyle\int_{\sigma_0}^{\sigma_1} 2\sqrt{\varphi_0 + h}\, d\sigma + \int_{\tau_0}^{\tau_1} dt$. It is easy to show that the condition $\delta A = 0$ is fulfilled for the actual path described by the system. In fact, putting $q_1 = f_1(\lambda), \cdots, q_k = f_k(\lambda)$, we can express A as a function of the parameter λ as follows:

$$A = \int_{\lambda_0}^{\lambda_1} \{2\sqrt{\varphi_0 + h}\sqrt{\Sigma a_{ij} \dot{q}_i \dot{q}_j} + \Sigma b_i \dot{q}_i\} d\lambda,$$

where we have put $\dot{q}_i = \dfrac{dq_i}{d\lambda}$.

Performing the variation of A, and remembering that the coordinates are given for $\lambda = \lambda_0$ and $\lambda = \lambda_1$, we obtain after an integration by parts and re-introduction of the variable t

$$\delta A = \int_{\tau_0}^{\tau_1} \Sigma L_i \delta q_i dt.$$

The coefficients L_i present themselves in the form

$$L_i = \frac{d}{dt}\left(\frac{\partial(T+W)}{\partial q_i'}\right) - \frac{\partial}{\partial q_i}(T+W).$$

They vanish in accordance with our hypothesis, and it is

thus proved, that the condition $\delta A = 0$ is fulfilled for the case under consideration.

In the theory of elliptic functions certain linear differential equations prove to be of importance, the coefficients of which are doubly periodic, while the integrals can also be represented as doubly periodic functions. In particular, two equations have been investigated, Lamé's and Picard's differential equations of the second order. In previous papers Professor Krause established a system of differential equations as an analogue of Lamé's equation. In his present paper he exhibits as the simplest analogue of Picard's equation a system of two linear differential equations, the coefficients and also the integrals of which are quadruply periodic functions.

Professor Shaw's paper is a complete generalization of quaternions to an algebra of n units. This is rendered possible by considering what has heretofore been called a *product* of two multiple numbers as simply a *function* of two such numbers, or of two sets of n variables. By the introduction of the symbol D defined by

$$D\bar{\alpha}\beta = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n,$$

where
$$\alpha = \sum_1^n a_i e_i, \quad \beta = \sum_1^n b_i e_i,$$

the e's being defining units; and by the introduction of a set of A's so defined that

$$A.a_1 a_2 \cdots a_r A.\beta_1 \cdots \beta_{r+1} = \begin{vmatrix} \beta_1 & \cdots & \beta_{r+1} \\ D\bar{a}_1\beta_1 & \cdots & D\bar{a}_1\beta_{r+1} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ D\bar{a}_r\beta_1 & \cdots & D\bar{a}_r\beta_{r+1} \end{vmatrix},$$

every quaternion formula finds its extension. The paper includes the theory of matrices of order n as the linear vector operators of such algebras. It is shown that every such matrix may be written in the form $\varphi = \sum_{1,1}^{n,n} a_{rs} v_{01}^{\ s} v_{10}^{\ r}$, where v_{01} and v_{10} are matrices of order n, $v_{01}^{\ n} = 1$, $v_{10}^{\ n} = 1$, $v_{01}^{\ s} v_{10}^{\ r} = i^{-rs} v_{10}^{\ r} v_{10}^{\ s}$, i being a primitive nth root of unity. Just as in quaternions $(1 + \sqrt{-1}\,k)i = i + \sqrt{-1}\,j$ is a nullity, so in matrices in general $\theta_{r-1} = (1 - v_{10})(1 - iv_{10})(1 - i^2 v_{10}) \cdots (1 - i^{r-2} v_{10})(1 - i^r v_{10}) \cdots (1 - i^{n-1} v_{10}) v_{01}$ is a

nullity, *i. e.*, $\theta_{r-1}^{2} = 0$. Any expression $\theta = \theta_a + \theta_b + \cdots + \theta_f$ is also a nullity, *i. e.*, $\theta^p = 0$, p being an integer less than n. A complete list of expressions for the moduli, etc., of matrices and of combined matrices is given.

Professor Davis presented the following graphic method for the solution of a system of linear equations. Let (1), (2), (3), (4) stand for four equations in x, y, z, w; let (12) stand for the result of eliminating z, and (12)′ for the result of eliminating w from (1) and (2); while (123) is the result of eliminating both z and w from (1), (2), and (3). Further, let $(1)_{ab}$ be the graph in the xy-plane of the result of putting $z = a$ and $w = b$ in (1), while $(12)_a$ is the graph of the result of putting $w = a$ in (12). To solve the equations draw $(1)_{00}$ and $(2)_{00}$, $(1)_{10}$ and $(2)_{10}$. Each pair meets upon $(12)_0$, which draw. Similarly determine and draw $(13)_0$ together with $(12)_1$ and $(13)_1$. Each pair meets upon the graph of (123). Draw this graph and likewise determine and draw the graph of (124). These two graphs meet in P whose coördinates are the proper values of x and y to satisfy the original equations. Finally w equals the distance from $(12)_0$ to P divided by the distance from $(12)_0$ to $(12)_1$, while the value of z is obtained by accenting the parentheses in the value of w. The method can be extended to any number of equations in as many unknowns.

The groups referred to in the title of Professor Dickson's paper are included in the author's list of simple groups published in the BULLETIN for July, 1899. The paper will appear in the *Transactions*.

Besides fundamental generalities (cf. BULLETIN, December, 1899, p. 99), the second paper of Professor Moore, which is to appear in the *Transactions*, develops certain examples of which the most important is the following: The group $F_{2,R,1}$ of binary fractional matrices $\left(\begin{matrix} a, & b \\ c, & d \end{matrix}\right)$, $(ad - bc = 1)$, of determinant 1 in any field or realm R of rationality, has the generators

$$T = \left(\begin{matrix} 0, & -1 \\ 1, & 0 \end{matrix}\right), \quad S_s = \left(\begin{matrix} 1, & s \\ 0, & 1 \end{matrix}\right),$$

s being any mark of the realm R; and the corresponding abstract group is given in the abstract generator symbols T, S_s, by the system of generational relations :*

* Another generational determination of this group was given at this meeting in the paper of Mr. J. A. Smith.

$$(1)\quad T^2 = I,\quad (2)\quad S_0 = I,\quad (3)\quad S_{\varepsilon_1} S_{\varepsilon_2} = S_{\varepsilon_1+\varepsilon_2},$$
$$(4)\quad S_{\varepsilon_1} T S_{\varepsilon_2} T S_{\varepsilon_3} T S_{\varepsilon_4} T S_{\varepsilon_5} T = I,$$

where I is the identity element, and in (3) ε_1, ε_2 are any two marks of the realm R, and in (4) ε_1, ε_2 are any two non-reciprocal marks of the realm R, and

$$\varepsilon_3 = \frac{\varepsilon_1 - 1}{\varepsilon_1\varepsilon_2 - 1},\quad \varepsilon_4 = -(\varepsilon_1\varepsilon_2 - 1),\quad \varepsilon_5 = \frac{\varepsilon_2 - 1}{\varepsilon_1\varepsilon_2 - 1}.$$

It should be noted that the marks $\varepsilon_1, \cdots, \varepsilon_5$ are the roots of a quintic equation whose coefficients are rational functions with integral coefficients of two arbitrary parameters and which in the realm of its coefficients is solvable by radicals.

Professor Newson investigates the geometrical properties of the group of 216 collineations in the plane and its subgroups with respect to a pencil of cubic curves remaining invariant under the group. In particular he discusses the effect of the transformations of the different subgroups on the harmonic and equianharmonic cubics and the four inflexional triangles contained in the pencil and their relation to the invariant triangles belonging to the transformations. The structure of the group is analogous to that of the tetrahedron group, the four equianharmonic cubics, the four degenerate cubics, and the six harmonic cubics of the pencil being analogous to the four vertices, four faces, and six edges of the tetrahedron.

Mr. Smith's paper exhibits in the first place certain novel generators of linear substitution groups which may be either homogeneous and integral in two letters or fractional in one. The determinant is taken as unity and the parameters are taken either 1° integers modulo q, 2° marks of a Galois field of order q^n together with integers modulo q, or 3° continuous variables either real or complex. For each of these three fields two types of generating substitutions S_α and V_α are defined, and relations are found which suffice for complete definition of the isomorphic abstract groups. These generators are so defined in each of the fields above mentioned that the formal relations are the same for all, the fractional linear group requiring three relations, the integral only two. In the second part of the paper normal representations of all elements of the groups are given in terms of generating elements, and their periodicity is determined by means of these normal forms.

In Professor Davis's second paper it is pointed out that the 48 operations which bring the cube into coincidence with itself are operations which bring any one of the eight triangles into which a face is divided by its lines of symmetry into coincidence with the 48 like triangles of the surface of the cube. Any of the subgroups of operations brings this same triangle into coincidence with merely a part of the 48 triangles. A cube with these triangles all of one color and the rest of the cube another color is an object which, taking account of color, comes into coincidence with itself only under the operations of the subgroup. Similar methods can be used for the representation of all regular groups and their subgroups.

THOMAS F. HOLGATE,
Secretary of the Section.

ON THE HISTORY OF THE EXTENSIONS OF THE CALCULUS.

BY J. G. HAGEN, S. J.

THE results attained by means of the infinitesimal calculus naturally evoked similar attempts in other directions. We may distinguish between two kinds of new theories, differing from each other mainly in their origin. The one is the natural outcome of an ever recurring need of solving practical problems or of giving existing theories a broader basis; the other owes its origin to purely abstract speculation. The former is not the property of any one mathematician; its nomenclature is not arbitrarily chosen, and its general introduction is only a question of time. This is clear from the history of all those branches of mathematics that have come into general use.

The case is somewhat different with the purely speculative theories. But though these may never succeed in demonstrating their practical utility, yet they are of value to mathematical science since they throw new light on existing methods and on their usefulness. For our purpose, however, a summary outline of these theories will suffice.

The present abstract is confined to those theories that are in close relation to the infinitesimal calculus and the theory of functions, and excludes, for instance, all geometrical methods and what are called "principles" or methods of demonstration.

In enumerating the following theories it cannot be the intention of the writer to classify them with the abstract theories that will never command general attention. They are mentioned merely because they have not yet been generally adopted. How long a new theory may remain a hidden treasure is amply illustrated by Grassmann's Ausdehnungslehre. Perhaps the reason why some of the theories considered have appeared of slight value is that they are formal in their nature and were invented before the full import and difficulty of Taylor's theorem was revealed. We expressly exclude from this remark Cauchy's calculus of residuals and the calculus of finite differences. How far the "operations" of this latter calculus may be promoted by the modern theory of functions is a question of great interest, which has probably not yet been fully answered.

While some of these new theories that have a set of formulas similar to the infinitesimal calculus are generally known, a synoptical view of all of them may be instructive. The references to the original sources would be be very numerous and cumbersome and are omitted here, as they will be given elsewhere.*

I. *Derivation with a General Index.*

The attempts to invent a calculus, of which the differential and integral calculuses should be particular branches, date back as far as Leibniz. Their history was outlined by Liouville, Tardy, and Cantor. That a negative index of differentiation is equivalent to an integration, and *vice versa*, was first stated by Leibniz. The first examples, in which the index n of an n^{th} differential coefficient is put equal to $\frac{1}{2}$, were given by Leibniz and Euler. Later, Fourier expressed a differential coefficient by a double integral, and added the passing remark that the index of the differential may be any number. While these few examples were intended as mere curiosities, the theory of the new calculus was established by Liouville and has since remained substantially in the shape that he gave it, although it was perfected later by Riemann, Gruenwald, Most, Letnikoff, and Krug.

Liouville gave three independent definitions of the derivation with a general index: first by development into a series, then by finite differences, and finally by definite integrals.

1. Liouville supposed the given function to be developed into an *exponential series*, of which he took the n^{th} differen-

*The writer's Synopsis of higher mathematics, vol. III., pp. 85–99.

tial, and then assumed the index n to be any number. His hypothesis is generally considered unsatisfactory. Riemann, while yet a student, chose for the same purpose a series progressing according to *powers* of the increment h in the development of $f(x+h)$, such that the exponents increase by integral numbers. The n^{th} derivation is then the coefficient of $k_n h^n$, where the factors k_n are subject to the condition, that the development becomes Taylor's theorem when n is an integer.

The definition by series has been abandoned by later writers on the subject.

2. A second definition was given by Liouville in the following manner: He expressed the *finite difference* $\Delta^n f$ by Newton's formula, then divided it by h^n, and finally took the limit for $h = 0$. If then the index n is taken arbitrarily, we have a definition for the general derivation. Gruenwald was probably the first to assume this index to be a complex number. He also expressly added a lower and upper limit to this derivation, as is required in a calculus which is to comprise the integral calculus as a special branch, and which may be valid only within definite limits in the plane of complex coördinates. Gruenwald's *notation* of the n^{th} derivation of $f(x)$ within the interval from a to x seems to be the most convenient: $D^n[f(x)]_a^x$.

3. A third definition is that by *definite integrals*. Liouville gave two integrals with the limits zero and infinity, and all later definitions are only generalizations of these. Gruenwald, as before mentioned, changed the limits into a and x. Most gave four different integrals, in each of which the upper limit is a critical point of the function $f(x)$. Letnikoff's two integrals resemble those of Liouville, except for the upper limit x. Krug's treatise on the same subject seems to surpass all others by a more precise statement of the conditions to which the formulas are subject. One of his integrals is a curve integral, and reminds us of Cauchy's fundamental theorem; the other has the limits a and z, and comprises all the cases discussed by Gruenwald and Most.

It will not be necessary for the purpose of this abstract to mention the various theorems which Liouville has established for the operation $D^n[f(x)]_a^x$.

II. *Cauchy's "Calcul des Résidues."*

This calculus is entirely due to Cauchy as to its origin and complete development. Unfortunately this calculus in

its entirety has not yet gained general recognition. Bertrand's "Calcul différentiel" is probably the only textbook that gives a complete theory of it. Of course, as far as this calculus is connected with Cauchy's fundamental theorem of integration around contours, it will ever remain a corner stone in the theory of functions.

1. The "Calcul des résidues" is not a generalization of the infinitesimal calculus, like the derivation with general index, but rather a branch parallel to it. Cauchy was probably led to this new idea by Lagrange's definition of the derivative as coefficient of the increment e^{+1} in the development of the function $f(z+e)$. For he defined the residuum as the coefficient of e^{-1}. This applies of course only to functions which have points of infinity. Suppose the equation $f(z)=\infty$ has the root z_r m times; then the function $(z-z_r)^m f(z)=\varphi(z)$ is in the point $z=z_r$ neither zero nor infinite. Hence, putting $z=z_r+e$, we have the development

$$f(z_r+e)=\frac{\varphi(z_r+e)}{e^m}=\frac{1}{e^m}\varphi(z_r)+\cdots+\frac{1}{e}\frac{\varphi^{(m-1)}(z_r)}{1.2\cdots(m-1)}$$
$$+\frac{\varphi^{(m)}(z_r+\theta e)}{1.2\cdots m}, \qquad (\theta<\pm 1).$$

Here the coefficient of $\frac{1}{e}$ is called the *residuum* of $f(z)$ in regard to the m-fold root z_r:

$$\frac{\varphi^{(m-1)}(z_r)}{(m-1)!}\equiv\frac{1}{(m-1)!}\frac{d^{m-1}}{de^{m-1}}\Big[\, e^m f(z_r+e)\,\Big]_{e=0}$$

2. The "*integral residuum*" of the same function $f(z)$ is defined as the sum of the residuals in regard to all the roots of $f(z)=\infty$ within certain limits, *i. e.*, within $z=a+iA$ and $z=b+iB$, and is designated by

$${}_a^b E_A^B((f)) \quad \text{or} \quad {}_a^b E_A^B[f],$$

where the letter E is the initial of the word "Extraction." It was changed by Cauchy into a character somewhat similar to a capital epsilon.

The "*partial residuum*" refers to a function composed of factors $f_1, f_2, \cdots$, and is designated by enclosing in brackets that factor of which the residuum is taken. Thus, $E[f_1]f_2\cdots$ refers to the roots of $f_1=\infty$ alone.

The sign E follows all the rules of the sign of integration, so far as the associative, the distributive, and the commutative laws are concerned.

Cauchy has established many theorems of this calculus which it is not necessary to mention here. By numerous applications to nearly all branches of mathematics and physics he has proved the great power of his calculus.

III. *Schell's "Quotial and Instaural."*

This interesting but little known calculus is similar to Cauchy's "Calcul des résidues," in as much as it branches off from the definition of the differential in a new direction. But while Cauchy started from Lagrange's idea of the derivative, Schell takes Leibniz's and Euler's point of view. His original idea is to replace the limit of a finite *difference* by the limit of a finite *quotient*. He endeavored to establish a kind of dualism in the infinitesimal calculus by interchanging differences and quotients, sums and products, somewhat similar to Gergonne's and Poncelet's dualism in geometry, where line and point coördinates are interchanged.

There are only two other publications relating to this subject; the one by Walton on the symbolic operator $\left(x\frac{d}{dx}\right)^r$, the other by Gaussin on the "Definition du calcul quotientiel." Gaussin considers this calculus mainly from the arithmetical and algebraic point of view.

1. The definition of the "quotial" consists in this: The finite difference $\Delta x = x_1 - x$ and the differential $\lim_{x_1=x} \Delta x = dx$ are replaced by the quotient $\theta x = x_1 : x$ and the "quotial" $\lim_{x_1=x} \theta x = \vartheta x$ (or $= qx$ in Gaussin's designation). Hence $x_1 = x + \Delta x$ has the dualistic formula $x_1 = x\theta x$. For the "quotial" of a function we have

$$u = F(x),\ u_1 = F(x\theta x),\ \vartheta u = \lim_{u_1=u} \frac{u_1}{u} = \frac{F(x\vartheta x)}{F(x)}.$$

Now, while in the differential calculus we pass from a differential to a differential quotient, we here pass from a "quotial" to a difference, by means of the logarithm. The peculiarity of this logarithm lies in the variable base θx, with the limit $\vartheta x = \omega = 1$. It is denoted by λ, and we have the following definition:

$$\textit{Quotial Logarithm of } F(x) = \lambda\vartheta F(x) = \lambda\frac{F(x\vartheta x)}{F(x)}.$$

The connection between this and the differential calculus appears from the following transformation :

$$\lambda\vartheta F(x) = \frac{x}{F(x)}\frac{dF(x)}{dx} = x\frac{d}{dx}\lg F(x).$$

which shows that the "quotial derivative" is substantially identical with a logarithmic differentiation.

On the other hand, it appears that "quotial derivatives" of higher order with the symbol

$$\lambda\vartheta^r \equiv \left(x\frac{d}{dx}\lg\right)^r$$

are not quite identical with Walton's operator $\left(x\frac{d}{dx}\right)^r$.

2. The "instaural," as the limit of a product, is the dualistic counterpart of the integral, the limit of a sum. We first write the quotial derivative in this form

$$\lambda\vartheta F(x) = \lambda\frac{F(x\omega)}{F(x)} = f(x), \text{ or } \frac{F(x\omega)}{F(x)} = \omega^{f(x)} \quad (\lim\omega = 1).$$

Then we divide the quotient of two given limits x_0 and x into n arbitrary factors

$$\frac{x}{x_0} = \omega_1 \cdots \omega_n,$$

and substitute the n values

$$x_0, \quad x_1 = x_0\omega_1, \quad x_2 = x_1\omega_2, \quad \cdots, \quad x_{n-1} = x_{n-2}\omega_{n-1},$$

in the above formula (putting at the same time for ω successively $\omega_1, \cdots, \omega_n$). Finally, multiplying all these equations, and remembering that $\omega_{r+1} = \frac{x_{r+1}}{x_r} = \vartheta x_r$, we have the product

$$\frac{F(x)}{F(x_0)} = \omega_1^{f(x_0)} \cdots \omega_n^{f(x_{n-1})} = P\omega_{r+1}^{f(x_r)} = P\vartheta x_r^{f(x_r)}.$$

If this product has a definite finite limit, it is called the "instaural of $f(x)$ between x_0 and x ("produit" according to Gaussin), and designated by

$$\frac{F(x)}{F(x_0)} = \prod_{x_0}^{x}\vartheta\sigma^{f(\sigma)}, \quad \text{or} \quad P\vartheta\sigma^{f(\sigma)} = \text{const. } F(x).$$

The connection between this formula and the integral calculus appears from the following identities:

$$f(x) = x\frac{d}{dx}\lg F(x), \quad \text{or} \quad C\cdot\exp\int\frac{f(x)}{x}dx = F(x),$$

$$\prod_{x_0}^{x}\theta\sigma^{f(\sigma)} = e^{\int_{x_0}^{x}\frac{f(\sigma)}{\sigma}d\sigma}$$

IV. *The Exponential Function of Higher Order.*

In considering the product as a repeated sum, and the power as a repeated product, it was but natural to look for a new expression which would repeat the power, and thus increase the number of elementary operations from seven to eight. The attempt, however, failed, for the reason, that base and exponent are not commutative, as are terms and factors. Yet we meet in the older literature the expression $r^{r^{r^{\cdot^{\cdot^{\cdot}}}}}$. It is not known where it occurs first. Condorcet equated this power of higher order to a certain series, without giving any proof. Euler, in supplying this proof, gave a full discussion of the function, and found an interesting relation between it and the base of natural logarithms.

1. He puts $\beta = r^\alpha$, $\gamma = r^\beta$, $\delta = r^\gamma$, $\cdots$ and investigates the condition that the series α, β, γ, $\cdots$ converge toward a finite value ω. Evidently this condition is $r^\omega = \omega$.

Euler then finds that the maximum value of the root r which can satisfy this condition is $r_1 = e^{1/e} = 1.44\cdots$ and the corresponding value $\omega = e = 2.7\cdots$ For all values $r > r_1$ the limit $\omega = x + iy$ is imaginary, and for all values $r < r_1$ the condition $r^\omega = \omega$ furnishes generally two values ω and ω', towards which the series α, β, γ, $\cdots$ can converge.

2. Assuming the root $r = e$ and writing x for α the function may be expressed by the symbol $e^{e^{\cdot^{\cdot^{\cdot^{x}}}}} = \exp^n x$, where "exp" is Cayley's notation for the exponential, and where n indicates the number of times the letter e is written. In this case the differential of the function can be written in the following manner:

$$\frac{d}{dx}\exp^n x = \exp^n x\cdot\exp^{n-1}x\cdots\exp x.$$

3. There are later essays on this function by Eisenstein and Woepke, which give no reference to Euler's treatise. They do not seem to have substantially advanced our knowledge of the nature of this function.

V. *Extension of the Calculus of Finite Differences.*

There are two very interesting new theories in the line of finite differences: one by Oettinger and the other by Dr. McClintock.

1. Oettinger creates a calculus perfectly analogous to the calculus of finite differences, another imitation of the geometric dualism. He runs side by side the two formulas

$$\Delta f(x) = f(x+h) - f(x) \quad \text{and} \quad \zeta f(x) = f(x+h) + f(x),$$

and all their consecutive corollaries. The original idea is the change of the *minus* in the former formula into *plus* in the latter.

The letter ζ (instead of Δ) is the initial of the word "Zunahme" (increase). Oettinger then develops a full theory of $\zeta^{\pm m}$ parallel to that of $\Delta^{\pm m}$, with many formulas and applications. This calculus does not seem to occur in the later literature.

2. Dr. McClintock has laid down his new calculus in an elaborate essay, which appeared in an early volume of the *American Journal of Mathematics*. He starts from the following symbol, where D is used for $\frac{d}{dx}$ as usual,

$$E^h\varphi(x) \equiv e^{hD}\varphi(x) \equiv (1+\Delta)\varphi(x) \equiv \varphi(x+h), \ \Delta x = h,$$

and designates the operation "E" as "enlargement."

A number of symbolic formulas with the operator E is then established, of which the following is remarkable for its simplicity, and seems to be the most important of all:

$$D \equiv \lg E.$$

Dr. McClintock then develops the theory of the functions of E, and uses it to give the theory of differentiation a broader basis, as the logarithmic branch of the doctrine of these functions.

It would be too long in this abstract to show how the formulas of the differential calculus are derived from the symbol $D = \lg E$. Nor can we do more than mention the remarkable application of this new method for finding simultaneously all the roots of an equation. It is hoped that this calculus of enlargement will give a new impulse to the use of symbolic operations which have proved so powerful in the hands of French and English mathematicians.

VI. *The Logarithmic Methods of Bergbohm and Oltramare.*

The facility with which exponentials can be handled in differentiating and integrating seems to have given rise to the two methods to be mentioned here. Both have this in common, that they pass from a given function to a logarithm or exponential, and, after a series of suitable transformations, back to the given function. This process appears almost like a roundabout way, by which the direct differentiation and integration is partly covered. Both methods were represented by their authors in a rather obscure and unsatisfactory manner.

1. The "new methods" of Bergbohm are based upon the symbol dx^{dy} with vanishing base and exponent. This symbol is first developed into the converging series

$$dx^{dy} = e^{dy\lg dx} = 1 + \frac{dy\lg dx}{1} + \frac{(dy\lg dx)^2}{1.2} + \cdots,$$

from which the fundamental formula is obtained

$$dx^{dy} = 1 + dy\lg dx.$$

Of this formula the logarithm is taken, and then the antilogarithm or numerus, both in regard to a certain vanishing base $d\beta$. These operations are designated by λ and ν. In this manner the further formulas are derived

$$\frac{dy\lg dx}{\lg d\beta} = \lambda(1 + dy\lg dx), \quad \nu\frac{dy\lg dx}{\lg d\beta} = 1 + dy\lg dx,$$

since $\lambda dx.\lg d\beta = \lg dx$.

The application consists in this, that by the above mentioned transformations any expression df is changed into $f'dx$, and again any expression $f'dx$ into df. By the former process the derivative is obtained, by the latter the integral. The method is tested by some of the elementary formulas.

2. Oltramare's "Calcul de generalisation" rests on the symbol

$$Ge^{xu+yv+zw+\cdots} = \varphi(x, y, z, \cdots),$$

by which the operation with the "characteristic" G is said to be defined.

The *partial* generalization, which refers to u only, or to v only, etc., is denoted by G_u, G_v, $\cdots$; and a *repeated* generalization by G^2, G^3, etc. The *inverse* operation G^{-1} is called "degeneralisation."

The principal operations in this calculus consist in developing both members of the equation

$$Ge^{(x+a)u+(y+b)v+\cdots} = \varphi(x+a, y+b, \cdots)$$

according to powers of the increment a, and equating the coefficients of equal powers. By this process the following fundamental formula is found:

$$Ge^{xu+yv+\cdots}\psi(u, v, \cdots) = \psi(D_x, D_y, \cdots)\varphi(x, y, \cdots).$$

In the applications the function ψ has to be developed according to powers of $D_x = \frac{d}{dx}$, $D_y = \frac{d}{dy}$, ..., after which the differentiations are to be performed.

The idea of this calculus seems to be to replace a given function $\varphi(x, y, \cdots)$ by the exponential function $Ge^{xu+\cdots}$, and then, after some suitable transformations, to return from the exponential to the given function. This suggests the idea that G is an operator like Σ, although the author avoids saying so, perhaps owing to the objections which had been raised against Liouville's assumption, that every function can be developed into an exponential series. The work of Oltramare was recently reviewed by Professor Lovett in the BULLETIN.

GEORGETOWN UNIVERSITY,
May, 1900.

BURNSIDE'S THEORY OF GROUPS.

Theory of Groups of a Finite Order. By W. BURNSIDE,* M.A., F.R.S.; Professor of Mathematics at the Royal Naval College, Greenwich. Cambridge, The University Press, 1897. 8vo., xvi + 388 pp.

THIS work enjoys the distinction of being the first treatise on the theory of groups which does not consider the applica-

* The joint author of Burnside and Panton's "Theory of Equations" is W. S. Burnside, an Irishman, professor of mathematics and fellow of Trinity College, Dublin. He is not related to the author of the work here considered, who is of Scotch extraction and has not published any other book. He has, however, published a number of memoirs—most of which relate to the theory of groups—in the *Proceedings of the London Mathematical Society* and in the *Messenger of Mathematics*. It seemed desirable to mention these facts since the two authors—William Burnside and William Snow Burnside—are frequently confused. Cf. Netto's review, Schlömilch's *Zeitschrift*, vol. 44 (1899), p. 20.

tions. As may be inferred from the title, the author has practically confined his attention to the groups of a finite order. The known regions which are bounded by these restrictions are, at the present time, not too extensive to be described in one volume. Happily, our author is so familiar with these regions that he does not confine himself entirely to leading the reader by known roads to the many interesting objective points. He has pointed out many short routes as well as a number of new objects of interest.

The theory of groups of a finite order (under the form of substitution groups) was first developed as a branch of the theory of equations; and the works of Lagrange,* Abel, and Galois on the solution of equations have contributed most powerfully towards its early development. In recent years it has been pointed out by Jordan, Klein, Cayley, and others that this subject has extensive geometrical applications. This has led to the study of groups in the abstract; *i. e.*, independent of any particular mode of representation. The work before us aims to treat the subject in this modern spirit.

Nevertheless considerable space is devoted to substitution groups. This may possibly be partly due to the fact that the subject was first developed from the standpoint of substitution groups. The author states in the preface that it is done because, "in the present state of our knowledge, many results in pure theory are arrived at most readily by dealing with properties of substitution groups." The concepts substitutions and group are so closely related that the former seems a natural precursor of the latter; and the study of many of the group properties, such as the group of isomorphisms and the transformation of conjugate subgroups and operators, seems to lead naturally to substitution groups. A knowledge of substitution groups can thus be utilized very frequently in the study of the abstract group properties, even if it is not essential for the study of these properties.

A number of interesting theorems such as those which relate to primitivity, imprimitivity, transitivity, and intransitivity apply only to substitution groups. Many readers would regret to see these landmarks of early triumphs banished from the theory to which they led. To such readers the method followed by our author will be entirely satisfactory, while those who have confined their attention to the more abstract theories might have preferred to find fewer

* Pierpont, "Lagrange's place in the theory of substitutions," BULLETIN, 2d series, vol. 1 (1895), p. 196.

substitution group illustrations—especially in the early chapters of the work. The notation of substitution groups has, however, been excluded from the proofs and investigations of chapters II. to VII., in which most of the fundamental properties have been developed.

The value of the book is increased by a number of carefully selected examples. These examples are generally chosen with a view towards continuing or completing the discussions in the text rather than with a view towards illustrating the theoretical developments. Consequently many of them seem too difficult to be taken up at the first reading. Some of them have not been stated with sufficient care: *e. g.*, number 2 on page 10 is evidently not true when $m = 1$ and $n > 2$; and the well known quaternion group is an exception to number 4, page 195, which is referred to Dyck. It may be remarked that the group of order 12 which contains six operators of order four is an exception to Dyck's theorem (*Mathematische Annalen*, vol. 22, p. 101) from which the given example seems to have been obtained. The second part of example 2, page 89, cannot be proved, since it is easy to construct groups of the specified type which are not generated by two operators that satisfy the given conditions.

As the book is written for the beginner, "to introduce to the reader the main outlines of the theory of groups of finite order without any applications," the first three chapters are devoted to the most elementary matters—the explanation of the notation of substitutions, the definitions and elementary illustrations of the terms group, subgroup, direct product,* etc., and a few fundamental theorems. The theorem stated on page 22 to the effect that every group of a finite order can be represented as a regular substitution group is not due to Dyck, as stated in the table of contents. It was given explicitly by Capelli at an earlier date (*Giornale di Matematiche*, 1878, p. 45) and somewhat less explicitly by Jordan at a still earlier date (Traité des substitutions, 1870, p. 60). The symbol G/Γ used to represent the quotient group of G with respect to its self-conjugate subgroup Γ seems to be due to Jordan (*Bulletin de la Société Mathématique*, vol. 1 (1873), p. 46). It is certainly not due to Hölder as is stated on page 38.†

* This definition of direct product does not appear to be as simple as possible; cf. *Transactions*, vol. 1 (1900), no. 1, p. 66. It may be remarked that Burkhardt gives an incomplete definition of direct product in his article on "Endliche discrete Gruppen," *Encyk. der math. Wissenschaften*, vol. 1, p. 219.

† Netto makes the same error in his new Algebra, vol. 2, 1900, p. 343.

A number of minor inaccuracies, which the reader will readily discover, occur in these as well as in the later chapters. Such inaccuracies tend to confuse the beginner, who is so prone to study the book, instead of using it as a tool to study the subject. For instance, on page 16, line 4, it is stated that the order of the product of two commutative operators is the least common multiple of the orders of these operators. That this is not always true may be seen from each of the following facts: the product of the two commutative substitutions $(abcd)(efgh)$ and $(afch)(bgde)$ is of order two, and the product of an operator and its inverse is identity. The statement is evidently always true when the groups generated by the two commutative operators have only identity in common. This condition is sufficient but it is not necessary. The interchange of the substitutions represented by D and E, on page 19, example VII., is an error of less importance.*

Chapter IV. is devoted to abelian groups. This subject is so important that a more extensive exposition would doubtless have proved agreeable to a large number of the readers. Several of the most interesting theorems are given as examples at the end of the chapter; but most beginners will probably be unable to prove them within a reasonable length of time. After a few preliminaries the author proves that any abelian group of order p^α (p being any prime number) contains a set of independent generators and that the orders of these generators are invariants of the group. This is followed by some very interesting investigations in regard to the subgroups of abelian groups of order p^α. Unfortunately the treatment of this subject is not always as complete as might be desired and there seem to be a few inaccuracies. For instance the notation of the latter part of theorem III., p. 58, seems to me to imply that a group of type (5, 2) contains a subgroup of type (3, 3), which is evidently impossible.

In chapter V. we meet the El Dorado of the theory of groups,—the groups of order p^α, p being any prime number. Cauchy proved that every group whose order is divisible by p must contain one or more subgroups of order p. More than twenty-five years later Sylow proved that a group contains one and only one system of conjugate subgroups of order p^α whenever its order is divisible by p^α but not by $p^{\alpha+1}$. These results indicated the great importance of a careful study of the groups of order p^α. Fortunately they give rise

* Mr. Kuhn called my attention to this interchange of substitutions.

to a large number of very interesting and unusually simple theorems. Many of these are old, but a considerable part of them are of recent origin being largely due to the investigations of Frobenius and Burnside. The general theorem published in the *Messenger of Mathematics*, vol. 27, 1897, appeared too late to be included in the present work.

The explicit theorems of this chapter together with those which are brought out by the illustrative examples give a comprehensive view of the groups of order p^a. The examples are drawn, to a large extent, from the publications of Hölder, Young, and Cole and Glover. On pages 68 and 69, Burnside undertakes the determination of every group G of order p^{r+s} in which the subgroup H_1 which is composed of the self-conjugate operators is the cyclical group of order r, while G/H_1 is an abelian group of type (1, 1, ···, with s units). He concludes that the number of these groups is $2^{\frac{1}{2}s}$, which is evidently incorrect.*

Chapter VI. is devoted to Sylow's theorem and the important extension due to Frobenius. For abelian groups this theorem was proved in Chapter IV. and the developments of the present chapter are somewhat simplified thereby. The fundamental importance of the theorem is clearly brought out by a number of carefully selected examples. In one of these examples all the possible groups of order 24 are determined. All these groups had been determined somewhat earlier,† but no reference is given to the earlier publications on this subject. The reference on p. 105 in regard to the published abstract groups is incomplete since all the groups whose order is less than 48 were known.‡ In giving the extension of Sylow's theorem the author follows Frobenius quite closely.

The composition series of a group, which is considered in Chapter VII., plays a very important rôle not only in the theory of groups but also in the Galois theory of equations. The foundation for the developments along this line was laid by Jordan in his article published in *Liouville*, vol. 14 (1869), p. 139. The reference on p. 119 should be to this article, instead of to Jordan's Traité des substitutions Here Jordan proved the very important theorem that the factors of composition of a given group are invariant. Hölder has extended these results somewhat by proving that the factor groups are also invariant. Burnside follows Hölder's developments quite closely in the first part of the

* Mr. Fite called my attention to this error.
† *Quar. Jour. of Math.*, vol. 28 (1896), p. 274.
‡ *Ibid.*, p. 32.

chapter, and he terminates it with a few theorems on soluble groups followed by several examples. In his recent Algebra, vol. 2, 1900, Netto uses chief series (Hauptreihe) in a somewhat different sense from Burnside, hence his example, p. 338, does not substantiate the criticism which Netto makes in the foot-note on p. 337 in regard to Burnside's theorem given on p. 123 of the present work.

The next three chapters (VII.–X.) are devoted to the theory of substitution groups. Only a brief sketch of this theory could be given in the limited space, especially since a great part of it is devoted to the determination of all the possible substitution groups of given degrees. The errors which occur in these enumerations have been noted elsewhere.* On p. 189 an error which occurs in Jordan's Traité (p. 65) is pointed out without noting that Jordan had corrected the error himself soon after the appearance of his work. The beautiful theorems of Jordan in regard to the limit of transitivity, which were published in the same article† as the corrections mentioned, would probably have been more useful than the one given on p. 152.

The material of these chapters seems, in general, to have been wisely selected. The many illustrative examples enable the student to get a clear insight into the real meaning of the terms transitivity, intransitivity, primitivity, and imprimitivity. The construction of intransitive groups furnishes a good means to acquire a thorough working knowledge of quotient groups and isomorphisms. The references in these chapters are not as complete as those of most of the other chapters. In some of the illustrative examples only a brief outline is given, so that the reader has to supply a number of intermediate statements to complete the proofs. Corollaries II., on pp. 177 and 184, evidently do not apply to the primitive groups of a prime order.

The principle of isomorphisms is one of those thought saving developments upon which special stress has been laid in recent years. In chapter XI. the isomorphisms of a group with itself are considered. The beginning of this chapter is not auspicious, for in the second paragraph it is stated that a group of order 2 is the only one in which each operator corresponds to itself when we let the inverse of each operator of a group correspond to itself. This evident inaccuracy is, however, followed by one of the best and most interesting chapters of the entire volume. The devel-

* BULLETIN, vol. 5 (1899), p. 249.

† Jordan, *Bull. de la Soc. Math. de France*, vol. 1 (1873), p. 41. Cf., BULLETIN, vol. 4 (1897), p. 144.

opments would doubtless have been simplified by giving some other methods of making a group isomorphic with itself, but the entire subject matter is so new that a great degree of simplicity could scarcely be expected.

In this chapter Burnside introduces a very useful new term (holomorph) which enables him to employ an interesting theorem in regard to regular groups, due to Jordan,* in the study of the group of isomorphisms. It would perhaps have been better to define the holomorph of a regular group G as the substitution group which includes all the possible substitutions in the elements of G that are commutative with G instead of the abstract group which is simply isomorphic to this substitution group. The largest subgroup of the given substitution group that does not involve a given element is evidently simply isomorphic to the group of isomorphisms of G.† In article 171, on the group of isomorphisms of an abelian group of order p^n and type $(1, 1, \cdots,$ to n units), reference should have been given to Moore's earlier article bearing on the same subject‡ and the theorem that the symmetric group of n elements is a complete group, except when $n = 6$, should have been referred to the BULLETIN, vol. 1, p. 258, as well as to *Mathematische Annalen*, vol. 46, p. 345.

Chapters XII. and XIII. are devoted to the graphical representation of a group and to Cayley's color groups. Here we meet for the first time some groups of an infinite order. The first chapter is, to a great extent, a reproduction of Dyck's well known article which was published in the *Mathematische Annalen* in 1882, vol. 20. The first volume of the *American Journal of Mathematics* contains a brief note on color groups by Cayley. Recently Maschke published a long article on this subject in the same journal, vol. 18, 1896, pp. 156–188. Burnside gives only a brief sketch of these groups, devoting merely five pages to them. Chapter XIV. contains a very brief sketch of the linear group, which forms the subject matter of the greater part of Jordan's Traité des substitutions. In the analysis of the fractional linear groups Gierster's noted memoir, published in volume 15 of the *Mathematische Annalen*, is closely followed. Moore has called attention § to the fact that the proof given on page 339, to the

* Traité des substitutions, 1870, p. 60.

† On page 221, vol. 1, of the *Encyclopädie der mathematischen Wissenschaften*, it is stated that the group of isomorphisms of G consists of those substitutions in the same elements that are commutative with G. This is evidently incorrect; cf. BULLETIN, vol. 5 (1899), p. 245.

‡ Moore, BULLETIN, vol. 2 (1895), p. 33

§ Moore, *Math. Annalen*, vol. 51, p. 419.

effect that the group of isomorphisms of the group of order 16 which has four independent generators is simply isomorphic to the alternating group of degree 8, is incorrect; and that the proper references are lacking in regard to the developments on pp. 336–339 and p. 342.

In the final chapter (XV.) of the book the soluble and composite groups are studied. In speaking of this chapter the author says (p. 334): "if the results appear fragmentary, it must be remembered that this branch of the subject has only recently received attention; it should be regarded rather as a promising field of investigation than as one which is thoroughly explored." Sylow's proof that every group whose order is p^α is soluble was one of the first general results in this direction. Recently Frobenius succeeded in establishing an equally general and interesting theorem by proving that all the groups whose order is not divisible by the square of a prime number are soluble. A large number of theorems of less extensive application have recently been published. These are clearly set forth in the present chapter, which seems more replete with recent results and problems awaiting solution than any other.

In conclusion we would say that the book seems to us to be a strong one notwithstanding the fact that the inaccuracies are somewhat numerous.* We have endeavored to call attention to a sufficient number of these to put the reader on his guard not to accept the results without a careful examination of the proofs. It is to be hoped that the demand for the book will warrant a new and carefully revised edition. A work containing such a rich store of facts, and nothing but facts, would be exceedingly valuable. It should be remembered that some of the mathematical works which have had a great influence on the developments in certain directions have been greatly marred by errors, so that we should not allow the minor defects of the present work to blind us to its many merits.

The student of the theory of groups will find here a rich storehouse of material, and the investigator will find numerous suggestions in regard to problems which await solution and methods of attacking them. The very errors will doubtless serve as a starting point for many investigations, since there are few things that can furnish a stronger impetus to many beginners than the thought of correcting a noted mathematician. It is therefore to be hoped that the book will be a means to arouse "interest among English (speaking) math-

* The book contains no references to Italian sources, although Italy has contributed very materially towards the development of this subject.

ematicians in a branch of pure mathematics which becomes the more fascinating the more it is studied," and thus accomplish the author's object.*

G. A. MILLER.

CORNELL UNIVERSITY,
March, 1900.

SHORTER NOTICES.

Traité de Nomographie. Par MAURICE D'OCAGNE. Paris, Gauthier-Villars, 1899. ix + 480 pp.

GIVEN a function of several variables, to read off its approximate values for all values of the variables, by simple inspection of a diagram drawn once for all—such is the general problem the investigation of which has led to this attractive volume. The diagram is often provided with travelling parts. The whole apparatus is called an *abacus*.

The problem is primarily a practical one. The technical arts force upon us relations or laws. So soon as a law is of frequent occurrence, its abacus is desirable. Even where great accuracy is required, an abacus is useful for getting a first approximation: for instance in the calculation of annuities (Abaque de M. Prévot), or in the solution of Kepler's equation

$$a - e \sin a = \mu$$

(abaque de M. d'Ocagne).

The scientific question as to what laws are capable of such exhibition is reserved for the final chapter. The rest of the book consists of classified instances. These instances are of quite surprising range and power. Practically a new subject is sprung upon us, claiming to be useful in so many directions that it would strain the faculty of an institute of technology to review the book in full detail. Thus we find an abacus (again by M. Prévot) used in the construction of the chromatic harp, an abacus of the penetration of light from a lighthouse (M. Allard), of the march of troops (M. Goedseels), of the deviation of the compass for an assigned ship (M. Lallemand), of the gauging of yachts (M. Chancel). These few random instances will serve to indicate the possibilities of the abacus in abolishing needless arithmetic. But the applications to the classic problem of

* P. vi.

cuttings and embankments must be especially mentioned, as M. d'Ocagne implies that the subject really grew out of this problem.

In the construction of an abacus, free use is made of the elements of analytic geometry, and in particular of both point and line coördinates. But the curve of the geometer is replaced at the outset by the scale (échelle). That is, the relation $y = f(x)$ is shown not by a curve but by the Y-axis itself, the point of this axis whose distance from the origin is $f(x)$ being marked simply x. Thus the common slide rule is built of two parts, either of which is a scale of logarithms, representing in the manner stated $y = \log_{10} x$.

So a scale of cosines might bear the numbers 0, 10, 20, ..., 180, the number x being distant from the origin $\cos x^0$, or $l \cos x^0$ if $2l$ is the length of the scale.

Such scales play a great part in the construction of abaci. But referring to the book itself for the other means employed, we shall do better to give a specific instance of an abacus.

Consider two points of a sphere, whose latitudes are λ and λ' and the difference of whose longitudes is L. The angle at the center is φ where

$$\begin{aligned} 2 \cos \varphi = {} & (1 + \cos L) \cos (\lambda - \lambda') \\ & - (1 - \cos L) \cos (\lambda + \lambda'). \end{aligned} \tag{1}$$

Write $u = -l \cos (\lambda + \lambda')$, $v = l \cos (\lambda - \lambda')$.

Then $$u(1 - \cos L) + v(1 + \cos L) = 2 l \cos \varphi. \tag{2}$$

Take a square of side $2l$, whose center is the origin. Mark a scale of cosines along the right side upwards, along the left side downwards, and along the top to the right. From the right scale read the number marked $\lambda - \lambda'$; its ordinate is u. From the left scale read the number $\lambda + \lambda'$; its ordinate is v. These numbers u and v are parallel coördinates of a line (as in Salmon, Conics, p. 386 of the sixth edition). And whatever values we assign to u and v, subject to (1), the line passes through a point; that is, (2) is the equation of a point. By an easy calculation this point is

$$x = l \cos L, \quad y = l \cos \varphi.$$

We lay then a line through the points $\lambda \pm \lambda'$, and observe where it cuts the vertical through the point of the top scale marked L; the ordinate of this point is $l \cos \varphi$, and φ itself is read off from the scale at the right.

The square with its scales, and the moveable line, constitute then the abacus of the equation (1). Naturally, the square is marked with vertical and horizontal lines to assist the reading.

The questions raised in the book should appeal not only to the technical man, but also to the teacher of elementary analytic geometry, at least to those teachers who care to heed that class of students whose cry is "of what use is this?"

FRANK MORLEY.

An Elementary Treatise on the Theory of Equations. By S. M. BARTON, Ph.D. D. C. Heath & Co., Boston, 1899. 8vo, xii + 198 pp.

THE title of Professor Barton's book suggests at once to the English-speaking student of mathematics the well-known treatises of Todhunter and of Burnside and Panton, and calls to mind how consistent English practice has been in assigning the theory of equations to distinctive works on the subject. Among the comparatively few books thus styled may be mentioned Chapman's "An Elementary Course in Theory of Equations," published in this country a few years ago.

The present treatise is much more limited in scope than the English works above mentioned. There is no attempt to deal with the formal side of the higher algebra. The book is intended expressly for undergraduate instruction in our colleges and technical institutions, and the contents and treatment are accordingly quite narrowly prescribed by the requirements of the usual college course.

The work falls into two parts: I., an elementary exposition of determinants; II., the theory of equations proper.

Part I.—The first two chapters give the principal theorems of determinants, and the third consists of applications to linear equations and a consideration of special determinant forms. The subject is introduced by considering the permutations of a group of elements, after which the development follows the usual course. The author here undoubtedly exposes himself to criticism for devoting so much space, some seventy-five pages in all, to a theory of which he is able to make very little use in the remaining chapters.

Part II.—In this part Burnside and Panton have been laid under heavy contribution. An inspection of the table of contents and a review of the pages show, as the author states, that the development has followed very closely in the lines of the first ten chapters of the treatise cited. It

will perhaps be fair to characterize this part as an attempt to adapt a portion of the Burnside and Panton to the needs of the general student, and at the same time to provide an introduction to the more extensive treatises on the subject. To secure the requisite condensation, an occasional change in arrangement has been made, statements and proofs abridged, and certain parts omitted. In all essential respects, however, the proofs, method, and arrangement of the English book have been retained.

In so far as the excellent features of the large treatise are found reproduced in the smaller work, to that extent is this grade of text-book enriched. A decided gain is the retention of the purely algebraical methods of proof. There is undoubtedly a tendency, quite marked in some of our college text-books on algebra, to replace the algebraical proofs of certain theorems by the methods of the calculus. The gain in an elementary text-book by a substitution of this character is neither apparent nor real, and does not warrant so radical a departure from algebraical methods; on the contrary, the change is attended by a loss in simplicity and uniformity of development.

It is to be regretted that Professor Barton has not availed himself of the opportunity to improve upon Burnside and Panton in the matter of drawing a sharp distinction between the variable x and its absolute value. The need of a careful discrimination in this direction is particularly felt in the chapter on the properties of entire polynomials.

The condensation and abridgements previously referred to have not always been effected without loss. Occasionally indeed there are statements which, unqualified, are not true. Again, there is a failure at times to express with sufficient directness and simplicity the nature and purposes of the propositions. Attention may be here directed to several cases in which changes seem desirable. In the introduction to Part II., page 77, the double assumption that α and β are distinct roots and that α is not zero, is a rather unhappy conjunction of hypotheses. So in chapter VII., page 125, following the statement of the general theorem, the corollary: "Corollary I. Every root of an equation is a divisor, *whole or fractional*, of the absolute term of the equation" should be eliminated. As it stands, it is incorrect; and if restored to the form in which it appears in Burnside and Panton, it is void of significance. On page 162, the reference to the sign of the intermediary Sturm function $f_r(x)$ is superfluous, and, moreover, the function does not necessarily change sign when x passes through α.

For completeness of statement, we note that the introduction to the second part is followed by a chapter of sixteen pages on complex numbers, containing De Moivre's theorem and the trigonometric solution of the binomial equation. There is also a chapter of six pages on elimination, with illustrations of Euler's and Sylvester's methods. The volume concludes with the methods of approximating to the real roots of numerical equations. Each theorem of the text, as far as possible, is immediately supplemented by problems showing its applications and place in the theory.

In conclusion, there are many admirable features in the work which would make it in some respects a valuable adjunct to instruction, and it is to be hoped that in a second edition a careful revision and the elimination of the rather numerous typographical errors will bring these qualities more clearly to light and place it in a position of considerable usefulness.

JAMES MACLAY.

The Theory and Practice of Interpolation: Including mechanical quadrature and other important problems concerned with the tabular values of functions. With the requisite tables. By HERBERT L. RICE, M.S. The Nichols Press, Lynn, Mass. 1899. Cr. 8vo., 234 + ix pp.

THE theory and practice of interpolation is, in its essentials, based on the fact that in nearly all functions which arise in physical problems, a small change of the variable produces a small change of the function. The simplest case is the one in which we may consider the ratio of the two small changes as constant: in the language of the subject, the first differences are constant. When this assumption will not give sufficiently accurate results, we have to consider the difference of the first differences or the *second* differences, and even differences of higher order. Ultimately we neglect differences of some definite order and thus implicitly reduce the problem to the consideration of the values of a function which, between certain limits and to a given degree of accuracy, may be considered rational, integral and algebraic. Round this problem a mass of literature has grown up. The values of a function are required for certain values of the variable. To obtain these every time from a formula may be troublesome, or even impossible if no such formula is known; tables are therefore made giving the values of the function for certain values of the variable, and the subject teaches how we can thence

obtain an approximate value of the function for the given value of the variable, and conversely.

Mr. Rice, in this volume, appears to aim at placing before the reader all the formulæ which are likely to be required for computing purposes, rather than to give an account of the principles which underlie the methods used. He has developed many formulæ very fully so as to show how they may be applied to any particular case, and has illustrated their use by many numerical examples. The book is essentially a practical one and as such will doubtless be of great service to a student wishing to learn the methods in use, as well as to one who knows them but desires to have the results gathered together in one volume; and the tables at the end will greatly increase its value. To one interested in the subject from the mathematical point of view only, it will perhaps be found to be lacking in interest. But this separation of the theoretical from the practical may be necessary here if either is to be of service. From the practical side, the processes used carry their own justification, that is, the results appear to justify the means employed. The theorist would probably prefer to investigate their validity and determine their limitations. One of the chief problems for the latter is the consideration of the limits between which any of the functions which arise for consideration can be represented, either theoretically or usefully for computing purposes, by a rational integral function, and the degree of error committed in so doing.

There are three principal formulæ used—those of Newton, Stirling, and Bessel—which really amount to a development by Taylor's theorem, with the assumption that we can always stop at some particular term and neglect the remainder. The differences between the three formulæ simply consist in the manner in which the various orders of differences are grouped; the best formula for any particular purpose is generally that which gives the required accuracy with the least amount of calculation. All these details are fully set forth by the author, and the reader will have no difficulty in immediately finding for practical use anything he may require.

One or two details call for slight criticism from the point of view of accurate statement. On p. 40 a remark is made which would leave the reader to infer that interpolation may give more accurate results than direct calculations even when both methods are available; this, if true, at least requires explanation. Again the author uses the elliptic integrals (with a real variable) as examples of non-

integrable expressions when the modulus is near unity; this is misleading, as a transformation enables us in such a case to approximate as close as we wish to the numerical value of the integral. The writer of this notice has not tested any of the formulæ but they appear to be carefully worked out and the author seems to be familiar with the best methods of computation. ERNEST W. BROWN.

Vorlesungen über Geschichte der Trigonometrie. Von DR. A. VON BRAUNMÜHL. Erster Theil. Leipzig, B. G. Teubner, 1900. 260 pp.

VON BRAUNMÜHL'S history of trigonometry is to be in two parts. The volume before us is the first part, and carries the narrative down to the invention of logarithms. If the second half of the work rises to the level of the first, the book will at once take its place as the most complete and authoritative work on the history of trigonometry.

In the perusal of this volume we realize the rapid progress which has been made in the study of mathematical history. The "restitution of decayed intelligence" has been carried on with great diligence. Formerly the origin of trigonometry was ascribed to Hipparchus and Ptolemaeus; now the earliest attempts to establish relations between the sides and the angles of a right triangle can be traced back to a period as long before the time of Hipparchus, as Hipparchus was before our time. The Ahmes papyrus of about 2000 B. C. contains five problems involving such relations.

In Whewell's History of the inductive sciences we read of the "sterility of Arabian genius," but in recent years a closer acquaintance with Arabic research in mathematics forces us to the admission that the Arabs displayed considerable originality. Von Braunmühl makes it plain that in trigonometry much of what has been ascribed to Europeans of the fifteenth century was worked out at an earlier period by the Arabs. The author suggests that even our present knowledge may require considerable revision after Arabic manuscripts have been read, which at present, covered with the dust of centuries, lie untouched on the shelves of Spanish libraries.

The author has embodied in this volume a great deal of original research. He has brought to light the fact that the Greeks possessed a graphic method of solving spherical triangles which was extended by the Hindus and Arabs so as to become a fruitful aid in trigonometric computation. This

remained in use as late as the seventeenth century. Von Braunmühl traces the beginning of the graphic treatment of spherical triangles back to Anaximander, but suspects that these graphic processes are much older, that they were known to the Egyptian and possibly also to the Chaldean astronomers.

The graphic method of solving spherical triangles is the oldest trigonometric *method* known to us. Ptolemaeus gives graphic processes in which *sine* is used, but curiously enough, in all trigonometric computation he employs instead *the chord of double the arc*. This anomaly finds its explanation in the fact that the Greeks treated the graphic solutions and the numerical solutions apart from each other; it remained for the Hindus and Arabs to unite the two methods and to recognize the advantage, in all cases, of using half the chord in place of the whole chord.

We have noticed no errors of importance. On page 88 the date of Alcuin's birth is given as 736. The same date occurs in Felix Müller's *Zeittafeln*, but the correct date is uncertain; it is probably 735. Snellius's baptismal name is spelled on page 70 *Willebrod*, in other places it is given correctly as *Willebrord*.

FLORIAN CAJORI.

Histoire des Mathématiques. Par JACQUES BOYER. Illustrée de fac-similés de manuscrits et de portraits. Georges Carré et C. Naud, Éditeurs. Paris, 1900. 250 pp.

ON opening this book the reader is attracted by several facsimile reproductions from old mathematical books or documents and by a number of portraits of mathematicians. Thus he has before him a facsimile of part of the Egyptian Akhmim papyrus, of the title page of the Acta Eruditorum, of a page of Euclid's Elements, from a manuscript preserved in the Bibliothèque Nationale of Paris. There are nineteen portraits of mathematicians. The list comprises eleven Frenchmen, three Englishmen, two Germans, two Russians, and one Swiss. Two likenesses are of women, namely, of Mme. Du Chatelet and Mme. Kovalevski.

To write a general history of mathematics and confine it, as M. Boyer does, to the small compass of 247 pages is no easy task. Anything like completeness cannot be looked for. Perhaps all one can expect is that the information offered be accurate, that the broad movements in mathematical thought be brought before the reader and that the narrative be made attractive so as to invite more thorough study in larger treatises. In this last respect we think that

M. Boyer has done well; but in the other two his success is not always so evident.

As a rule the information is up to date and the facts are accurately presented. In the case of Mme. Kovalevski he puts us on our guard against the usual 1853 as the date of her birth, and gives it correctly as 1850. In some places the author fails to embody recent results of historical research. He mentions Argand on the graphic representation of the imaginary, but says nothing of Wessel; he does not mention Taurinus in connection with Gauss, Lobachevski and the Bolyais; the native place of Serenus is given as Antissa, but according to Heiberg it should be Antinoeia. M. Boyer refers to the question whether or not "the physical space of our experience" is that of Lobachevski and then says: "Peirce conclut par l'affirmative en se basant sur de délicates observations des parallaxes d'étoiles fixes." By the reference given in the alphabetical index this remark is attributed to Benjamin Peirce, but this is probably an error, for it was Mr. Charles S. Peirce who once maintained that he had demonstrated the above.

M. Boyer has failed to emphasize that, according to recent study of Arabic manuscripts, Nasir Eddin and other Arabs had really developed trigonometry to a much fuller extent than was formerly supposed. We were rather surprised to find no reference whatever to such original workers in algebra as D. F. Gregory, George Peacock, and George Boole, while an eighteenth century English algebraist of lesser note is not only mentioned but also honored with a portrait. The author has signally failed in setting before the reader a view of the development of theories. For instance, nothing worthy of notice is given on the growth of the theory of limits, or the gradual extension of the number concept. No matter how inadequate the space may be for recent history, it would seem as if the broad fact should be presented that at the beginning of the nineteenth century mathematics had no adequate logical foundation, and that Gauss, Abel, and Cauchy were compelled to begin to build anew on firmer ground. The alphabetical index is inaccurate and incomplete.

Notwithstanding these defects the book will be useful in attracting students to the study of the history of mathematics and making them acquainted with the names and some of the achievements of great mathematicians.

FLORIAN CAJORI.

Vorlesungen über Kreis- und Kugel-Functionen-Reihen. Von Dr. Johannes Frischauf. Leipzig, B. G. Teubner, 1897, pp. 60.

This little treatise will be welcomed by students of pure analysis and also by students of mathematical physics. The subject treated has already an extensive literature of its own; but the important results in this field are scattered through different memoirs of Dirichlet, Du Bois Reymond, Bonnet, Lipschitz, Dini, and others, while there hasappeared no distinctly elementary treatment of the subject as a whole. Thorough and comprehensive treatments of Fourier's series and other allied developments have indeed been given by C. Neumann in his "Uber die nach Kreis- Kugel- und Cylinder-Functionen fortschreitenden Entwickelungen," and by Dini in his "Serie di Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale," but these treatments could hardly be called elementary—certainly not in comparison with the above work. The author has followed a uniform and easy method throughout and has shown the convergence of Fourier's series, of Fourier's integral, and of the developments in terms of zonal and spherical harmonics in case the function to be developed has but a limited number of maxima and minima within a certain interval and is either continuous within this interval or else has but a finite number of so called "finite discontinuities." Some of the simpler cases are also considered in which the maxima and minima become infinite in number in the neighborhood of certain points, or in which the function at certain points becomes infinite in value. In brief, the author has given us an elementary, systematic treatment of these series for all cases which might ordinarily arise in connection with nature.

Chapter I. deals especially with Fourier's series. Chapters II. and III. are devoted to the discussion of certain general properties of zonal and spherical harmonics, and Chapter IV. relates to the developments in terms of the latter functions. Illustrative examples are given throughout.

W. B. Ford.

NOTES.

THE sessions of the coming summer meeting of the Society will be held on the afternoons of Wednesday, Thursday, and Friday, June 27th–29th, alternating with those of Section A of the American association for the advancement of science. A fourth session will be added if necessary. The place of meeting is room 506, Fayerweather Hall, Columbia University. The Council will meet on Thursday morning.

THE ninth conference of the German association for promoting instruction in the mathematical and natural sciences was held at Hamburg, June 4th–7th, 1900.

A NEW scientific journal, *Il Bolletino di matematiche e di scienze fisiche e naturali*, devoted to the interests of the teachers and students of the normal schools of Italy, has been lately founded at Bologna, under the editorship of Professor ALBERTO CONTÉ. The initial number appeared December 1, 1899, and each annual volume will consist of twenty-four numbers.

THE *Revue Philosophique* for April, 1900, contains two articles of interest to mathematicians: "La première antinomie mathématique de Kant," by C. DUNAN, and "l'Antinomie du transfini," by É. BOREL.

THE first and second numbers of the first volume of the third series of *Bibliotheca Mathematica* under the editorship of Gustaf Enestrŏm appeared from the press of B. G. Teubner April 30th, 1900. The enlarged sphere of this journal is amply indicated by the table of contents of this double number of two hundred and ninety-six pages, containing a portrait of Sophus Lie as frontispiece, some thirty papers, obituary notices of C. I. Gerhard, F. Rosenberger, H. E. Wappler, and L. G. Gascó, a chronological list of the writings of Sophus Lie by F. Engel, accounts of the annual meetings held by mathematicians in Germany, France, England, and America last summer, reviews, new publications, and current notes of scientific interest.

THE last three parts of GINO LORIA'S Le scienze esatte nell' antica Grecia are announced for early publication.

The first two parts appeared in 1893 and 1895. The concluding books are entitled respectively: Il substrato matematico della filosofia naturale dei Greci; Il periodo argenteo della geometria Greca; L'aritmetica dei Greci.

THE first part of FELIX MÜLLER'S Mathematisches Vocabularium in deutscher und französischer Sprache is in the press. The completed work will contain ten thousand terms of pure and applied mathematics; it has been prepared with the assistance of J. NEUBERG and A. WANGERIN, and represents years of labor. The authors hope that this vocabulary will prove to be 1° a useful supplement to all French and German dictionaries, 2° the forerunner of an exhaustive mathematical dictionary, 3° a lexicographical guide to the terminology of the mathematical sciences.

THE third volume of the second edition of M. CANTOR'S Vorlesungen über Geschichte der Mathematik is in the press; it will appear in three parts.

FELIX L. DAMES, of Berlin, announces the appearance of the first part of the third and final volume of Professor J. G. HAGEN'S Synopsis der höheren Mathematik. The three volumes are devoted respectively to algebraic analysis, geometry, and differential and integral calculus.

THE UNIVERSITY OF CHICAGO. The announcement for the summer quarter of 1900 appeared in the May number of the BULLETIN. During the three quarters (autumn, winter, spring), October, 1900, to June, 1901, the following mathematical courses will be offered, four or five hours weekly:—By Head Professor E. H. MOORE: Theory of groups with applications to algebra (introductory course followed by a seminar), autumn and winter quarters; Functions of a real variable, spring quarter; Advanced integral calculus I, autumn quarter; Advanced algebra II, winter quarter.—By Professor OSKAR BOLZA: Theory of elliptic functions (Weierstrass) with applications, winter and spring quarters; Calculus of variations, winter quarter; Theory of functions of a complex variable, spring quarter.—By Associate Professor H. MASCHKE: Selected chapters of algebra, autumn quarter; Projective geometry, autumn quarter; Modern analytic geometry (continuation), winter quarter; Advanced integral calculus II, winter quarter.—By Assistant Professor J. W. A. YOUNG: Theory of equations I, autumn quarter.—By Assistant Professor L. E. DICKSON: Lie's theory of differential equa-

tions, winter quarter; Continuous groups, spring quarter.—By Dr. H. E. SLAUGHT: Solid analytical geometry, spring quarter.—By Dr. J. H. BOYD: Theoretical mechanics, autumn quarter; Differential equations, spring quarter.—By Dr. F. R. MOULTON: Analytical mechanics I, II, autumn and winter quarters.

YALE UNIVERSITY. The following courses in mathematics are announced for the year 1900–1901:—By Professor J. WILLARD GIBBS: Vector analysis, three hours; Multiple algebra, two hours; Electromagnetic theory of light, two hours.—By Professor W. BEEBE: Celestial mechanics, three hours.—By Professor J. PIERPONT: Differential equations and theory of functions, three hours; Galois's theory, three hours;† Theory of numbers, three hours;† Partial differential equations and in particular those of dynamics, three hours.†—By Professor P. F. SMITH: Continuous groups, two hours; Transformations of space, two hours;† Differential geometry, two hours.†—By Professor G. P. STARKWEATHER: Theoretical mechanics (advanced), two hours.—By Dr. M. B. PORTER: Linear differential equations, three hours; Calculus (advanced), three hours.—By Dr. H. E. HAWKES: Modern geometry, three hours; Advanced algebra, three hours.†—By Mr. E. B. WILSON: Algebraic plane curves, three hours;† Theoretical mechanics (elementary), three hours.†

Courses marked † extend through only one-half of the year.

A MATHEMATICAL seminary, appointed as are those of the universities of Göttingen and Leipzig, has been equipped at the University of Jena.

THE mathematical prize of the Istituto Lombardo di scienze e lettere in Milano, will be awarded for the year 1901 for a study of the differential equations of electrotechnics that will indicate the most practicable methods for their approximate integration and illustrate the exposition with examples.

THE Istituto Veneto delle scienze, lettere, ed arti offers its prize for the year 1902 for a memoir on the projective character of two dimensional surfaces in space of n-dimensions.

PROFESSOR THOMAS CRAIG, of Johns Hopkins University, died at Baltimore, May 8th, 1900, aged 44 years. Professor

Craig was one of the pioneers in the great movement of the last quarter of a century which has brought American mathematicians in touch with continental ideas and methods. He graduated at Lafayette College in 1875, with the degree of civil engineer; but with the advent of Sylvester in America his natural interest in pure mathematics immediately asserted itself. At the opening of Johns Hopkins University in 1876, he was appointed to a fellowship and at the same time was entrusted with a part of the mathematical instruction. In 1878, he received the degree of doctor of philosophy from Johns Hopkins, his dissertation being entitled: "Representation of one surface upon another, and on some points in the theory of the curvature of surfaces." During the years 1879–1881, he was connected with the United States Coast Survey, continuing, however, to give certain lectures at the University. His connection with the University covered the entire twenty-four years of its existence. Since 1892 he held a full professorship of mathematics. A considerable part of his activity for twenty years was devoted to the *American Journal of Mathematics*, of which he was an editor from 1883, and editor-in-chief from 1893 to 1898. Besides numerous articles in the *American Journal of Mathematics* and elsewhere, Professor Craig published a treatise on the mathematical theory of projection (1879), a treatise on linear differential equations (1889), and two briefer works on fluid motion (1879).

PROFESSOR L. BIANCHI has been made a member of the editorial board of the *Annali di Matematica* in the place of the late E. Beltrami. The board now consists of L. Cremona, U. Dini, G. Jung, and L. Bianchi.

PROFESSORS W. FÖRSTER, of Berlin, and E. WEISS, of Vienna, recently celebrated twenty-five year jubilees as professors at the respective universities.

PROFESSOR L. BOLTZMANN, of Vienna, has accepted the call to a professorship of physics recently tendered him by the University of Leipzig.

MR. J. F. HUDSON, of Oxford University, has been appointed mathematical lecturer at University College, Bristol, in the place of Mr. J. F. MCKEAN, who resigned to become mathematical lecturer at the Royal Naval Engineering College, of Devonport.

PROFESSOR F. MORLEY, of Haverford College, has been appointed head of the department of mathematics at Johns Hopkins University.

AT Yale University the following changes have taken place in the mathematical staff: In the Sheffield Scientific School, assistant professor PERCEY F. SMITH has been promoted to a full professorship, and Dr. G. P. STARKWEATHER to an assistant professorship of applied mechanics. Mr. E. B. WILSON has been appointed instructor in the academic department.

AT Princeton University Mr. A. H. WILSON resumes his duties as instructor in mathematics after spending a year on leave at the University of Göttingen, and Dr. E. O. LOVETT has been promoted to a full professorship in mathematics and granted a year's leave of absence.

A SEPARATE department of astronomy has been organized at the University of California and placed under the charge of associate professor A. O. LEUSCHNER. DR. E. M. BLAKE and MR. D. N. LEHMER have been appointed instructors in the department of mathematics.

AT Columbia University, Mr. C. J. KEYSER, has been promoted to an instructorship in mathematics, and Mr. D. H. POLLARD has been appointed assistant in mathematics.

DR. G. H. LING, of Wesleyan University, has been appointed professor of mathematics at the University of Cincinnati for the summer session of 1900.

MR. A. H. LYBYER, of Princeton University, has been made professor of mathematics at Roberts College, Constantinople.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BRUNEL (G.). See ENCYKLOPÄDIE.

BURALI-FORTI (C.). Les propriétés formales des opérations algébriques. Turin, 1900. 12mo. 40 pp. Fr. 1.00

BURKHARDT (H.). See ENCYKLOPÄDIE.

CARNOY. Cours de géométrie analytique. Vol. I: Géométrie plane. 6e édition. Paris, Gauthier-Villars, 1899. 8vo. Fr. 11.00

COX (H.). Rudimentary treatise on integral calculus. New edition. London, Lockwood, 1900. 12mo. 136 pp. (Weale's series.) 1s. 6d.

DEHN (M.). Die Legendre'schen Sätze über die Winkelsumme im Dreieck. [Diss.] Leipzig, Teubner, 1900. 8vo. 38 pp. M. 1.20

ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Vol. II., redigiert von H. Burkhardt. Heft 2 und 3: G. Brunel, Bestimmte Integrale [Schluss]; P. Painlevé, Gewöhnliche Differentialgleichungen, Existenz der Lösungen; E. Vessiot, Gewöhnliche Differentialgleichungen, elementare Integrationsmethoden; E. von Weber, Partielle Differentialgleichungen. Leipzig, Teubner, 1900. 8vo. Pp. 161–399. M. 7.50

GECK (E.). Ueber die singulären Punkte algebraischer Flächen. [Diss.] Tübingen, Pietzcker. 1900. 8vo. 40pp. M. 1.00

GUTZMER (A.). See JAHRESBERICHT.

HAUCK (G.). See JAHRESBERICHT.

HENTSCHEL (O.). Ausführung einiger konformen Abbildungen. I. (Progr.) Salzwedel, 1899, 4to. 8 pp., 4 plates.

HERMITE (C.). See SERRET (J. A.).

HERFF (E.). Die Maxima und Minima einer verwandelbaren Funktion. (Progr.) Sigmaringen, 1899. 4to. 18 pp.

JAHRESBERICHT der deutschen Mathematiker-Vereinigung, enthaltend die Chronik der Vereinigung für das Jahr 1899 und die auf der Versammlung zu München gehaltenen Vorträge. Herausgegeben von G. Hauck und A. Gutzmer. Leipzig, Teubner, 1900. 8vo. 231 pp.

KINN (G. A.). Die Anwendung unendlicher Produkte in der Funktionentheorie. (Progr.) Sächsisch-Regen, 1899. 4to. 32 pp.

PAINLEVÉ (P.). See ENCYKLOPÄDIE.

PESCIONE (A.). Le coniche in coordinate cartesiane; appunti di geometria analitica. 1899. 8vo. Fr. 2.50

PICARD (E.) et SIMART (G.). Théorie des fonctions algébriques de deux variables indépendantes. Vol. II, fascicule 1. Paris, Gauthier-Villars, 1899. 8vo. 4 + 206 pp. Prix du volume complet Fr. 14.00

PRINCIPALES formules de la théorie des fonctions elliptiques; tableau résumé publié par les *Nouvelles annales de mathématiques*. Paris, Gauthier-Villars, 1900. 8vo. (Concours pour l'agrégation de mathématiques.) Fr. 0.75

RADELFINGER (F. G.). Linear differential equations. (*Philosophical Society of Washington, Bulletin*, Vol. 14, pp. 21–35.) Washington, 1900. 8vo.

SERRET (J. A.). Cours de calcul différentiel et intégral. 5e édition, augmentée d'une note sur les fonctions elliptiques, par C. Hermite. (En 2 volumes.) Vol. I : Calcul différentiel ; 13 + 617 pp. Vol. II: Calcul intégral ; 13 + 904 pp. Paris, Gauthier-Villars, 1900. 8vo. Fr. 25.00

SIMART (G.). See PICARD (E.).

SPOHER (B.). Zur Ableitung allgemeiner Eigenschaften algebraischer Kurven. (Progr.) Ehingen, 1899. 4to. 24 pp.

TORROJA (E.). Tratado de geometria de la posicion y sus aplicaciones á la geometria de la medida. Madrid, 1899. 4to. 813 pp. Fr. 25.00

VESSIOT (E.). See ENCYKLOPÄDIE.

VILLAFRAÑE Y VIÑALS (J. M.). Tratado de analisis matematica (algebra superior). 2a edicion, corregida y aumentada. Barcelona, 1899. 4to. 996 pp. Fr. 25.00

WEBER (E. VON). See ENCYKLOPÄDIE.

WEINER (F.). Eine Anwendung der Hermite'schen U-Funktionen. (Progr.) Wien, 1899. 8vo. 16 pp.

II. ELEMENTARY MATHEMATICS.

ALEU (M. L.). Elementos de matematica. Geometria. 2a edicion. Madrid, 1899. 4to. 8 + 268 pp. Fr. 6.50

ARNDT (E.). Hauptsätze der ebenen Geometrie, nebst Uebungsaufgaben zum Gebrauche an Volke- und Mittelschulen. 7te Auflage. Berlin, Oehmigke, 1900. 8vo. 4 + 56 pp. M. 0.50

CHRYSTAL (G.). Algebra. Elementary text-book for higher classes of secondary schools, etc. Part 2. 2nd edition. Edinburgh, Black, 1900. 8vo. 632 pp. 12s. 6d.

COMBEROUSSE (C. DE). See ROUCHÉ (E.).

CONSTABLE (W. G.) and MILLS (J.). Elementary algebra, including quadratic equations. London, Longmans, 1900. 12mo. 272 pp. Cloth. 2s.

F. (F.). Manuel d'algèbre et de trigonométrie. Paris, Poussielgue [1900]. 16mo. 8 + 216 pp.

FOCKE (M.) und KRASS (M.). Leitfaden zur Einführung in die Stereometrie und Trigonometrie. Sonderausgabe für die Untersekunda aus den gleichnamigen Lehrbüchern. 3te Auflage. Münster, Coppenrath, 1900. 8vo. 32pp. Boards. M. 0.60

HARMUTH (T.). Textgleichungen geometrischen Inhalts. Für den Gebrauch beim Unterricht entworfen. 2te Auflage. Berlin, Springer, 1900. 8vo. 4 + 75 pp. Boards. M. 1.20

KRASS (M.). See FOCKE (M.).

MILLS (J.). See CONSTABLE (W. G.).

RÖLLNER (F.). (*a*) Beweis eines Gesetzes der gleichzeitig gleichschnellen Rotationen. (*b*) Ueber Aehnlichkeit und Symmetrie als grundlegende Principien der Geometrie. (Progr.) Römerstadt, 1899. 8vo. 17 pp.

ROUCHÉ (E.) et COMBEROUSSE (C. DE). Traité de géométrie. 7e édition, revue et augmentée, par E. Rouché. Première partie: Géométrie plane. Paris, Gauthier-Villars, 1900. 8vo. 42 + 548 pp. Fr. 7.50

TESTI (G. M.). Nozioni di geometria ad uso più specialmente delle allieve dei corsi complementari (già scuole preparatorie normali). 4a edizione migliorata. Livorno, Giusti, 1900. 16mo. 8 + 100 pp. Fr. 1.00

WEISBACH (J.). Tafel der vielfachen Sinus und Cosinus, sowie der vielfachen Sinus versus von kleinen Winkeln, nebst Tafel der einfachen Tangenten. 6te Auflage. Berlin, Weidmann, 1900. 8vo. 28 pp. M. 1.00

III. APPLIED MATHEMATICS.

ALBRECHT (T.). Bericht über den Stand der Erforschung der Breitenvariation am Schlusse des Jahres 1899. Herausgegeben vom Centralbureau der internationalen Erdmessung. Berlin, Reimer, 1900. 4to. 26 pp., 1 plate. M. 3.00

BLAHA (E.). See LEIST (C.).

CHARBONNEL (J.). La loi du trapèze. Paris, Dunod, 1900. 8vo. 75 pp.

EGGERS (W.). Lehrbuch der darstellenden Geometrie. Teil I: Die Elemente der darstellenden Geometrie. 2te Auflage. Leipzig, Seemann, 1900. 8vo. 7 + 54 pp., 5 plates. Boards. M. 1.50

GUBATZ, LÜDEKE, und WEIGEL. 301 Aufgaben aus der darstellenden Geometrie für Maschinenbauer, Kesselschmiede und verwandte Gewerbe. Mit kurzen praktischen Lösungen und 333 Zeichnungen. Leipzig, Seemann, 1900. 8vo. 34 pp. Boards. M. 2.25

HALL (E. H.). Elementary lessons in physics; mechanics (including hydrostatics) and light. New York, Holt, 1900. 12mo. 10 + 182 pp. Cloth. $0.65

HELLER (J. F.). See SMOLIK (F.).

HOFF (J. H. VAN'T). Vorlesungen über theoretische und physikalische Chemie. Heft 3: Beziehungen zwischen Eigenschaften und Zusammensetzung. Braunschweig, Vieweg, 1900. 8vo. 10 + 36 pp. M. 4.00

JOHNSON (J. B.). Theory and practice of surveying; for surveyors and engineers generally, but especially for students of engineering. 15th edition, revised and enlarged. New York, Wiley, 1900. 8vo. 900 pp. Cloth. $4.00

LACHMANN (M.). See RODET (J.).

LEIST (C.). Die Steuerungen der Dampfmaschinen. Zugleich als vierte Auflage des gleichnamigen Werkes von E. Blaha. Berlin, Springer, 1900. 8vo. 15 + 770 pp. Cloth. M. 20.00

LÜDEKE. See GUBATZ.

MERCHICH (M.). Isagoge in principia mathematica oeconomices politicae, eruditis orbis catholici congressum quintum mense augusto anni 1900 Monaci celebraturis tradita. Vienna, Horvát-Kimle, Merchich, 1900. 8vo. 26 pp. M. 0.40

MONTCHOISY (DE). Cours pratique et théorique de machines à vapeur, professé à l'Ecole supérieure de maistrance de la marine. 3e édition, (En deux volumes.) Vol. I: 15 + 623 pp.; Vol. II: 671 pp. Paris. Challamel, 1900. 8vo.

NOVAT (J.). Cours pratique de résistance des matériaux, professé à la Société d'enseignement professionel du Rhône. Paris, Béranger, 1900. 18mo. 2 + 440 pp.

RITTER (W.). Anwendungen der graphischen Statik. Nach C. Culmann bearbeitet. Teil 3: Der kontinuierliche Balken. Zürich, Raustein, 1900. 8vo. 12 + 270 pp., 4 plates. M. 9.60

RODET (J.). Berechnung der Leitungen von Mehrphasenströmen. Uebersetzt von M. Lachmann. Leipzig, Leiner, 1900. 8vo. 7 + 55 pp. M. 2.75

SCHMIDT (J.). Anleitung zur Konstruktion von Sonnenuhren. (Progr.) Plan, 1899. 8vo. 21 pp., 3 plates.

SMOLIK (F.). Elemente der darstellenden Geometrie. Neu bearbeitet von J. F. Heller. 2te, vermehrte Auflage. Leipzig, 1900. 8vo. Cloth. M. 4.00

WEIGEL. See GUBATZ.